

THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc., F.R.S.
AND
PROF. E. T. WHITTAKER, Sc.D., F.R.S.

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CONTENTS.

	PAGE
S. RAMANUJAN. B. M. WILSON, M.A., D.Sc.,	89
SOME INEQUALITIES CONNECTED WITH A METHOD OF REPRESENTING POSITIVE INTEGERS. F. S. MACAULAY, D.Sc., F.R.S.,	95
THE PARTICULAR INTEGRALS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS, WITH SPECIAL REFERENCE TO A FORMULA GIVEN BY FORSYTH. F. UNDERWOOD, M.Sc.,	99
MULTIPLICATION AND DIVISION OF DECIMALS. R. A. M. KEARNEY, B.A.,	102
MATHEMATICAL NOTES. A. DE MORGAN, M.A.; C. FOX, D.Sc.; N. M. GIBBINS, M.A.; C. W. HILDEBRAND, B.A.; A. J. W. KEPPEL; L. J. ROGERS, F.R.S.; MISS E. J. TERNOUTH, M.A.; G. WOTHERSPOON, M.A.,	111
REVIEWS. T. A. A. BROADBENT, M.A.; N. M. GIBBINS, M.A.; E. R. HAMILTON, M.A.; H. LOB, M.A.; V. NAYLOR, M.Sc.; G. SMEAL, B.Sc.,	120
CORRESPONDENCE. HIGHER TRIGONOMETRY FOR SCHOOLS. A. W. SIDMONS, M.A.,	126
PERSONAL NOTES,	129
THE LIBRARY,	130
GLEANINGS FAR AND NEAR (745-767),	94
ERRATUM,	132
THE BRANCHES: BOOKS AND JOURNALS RECEIVED,	i-iv

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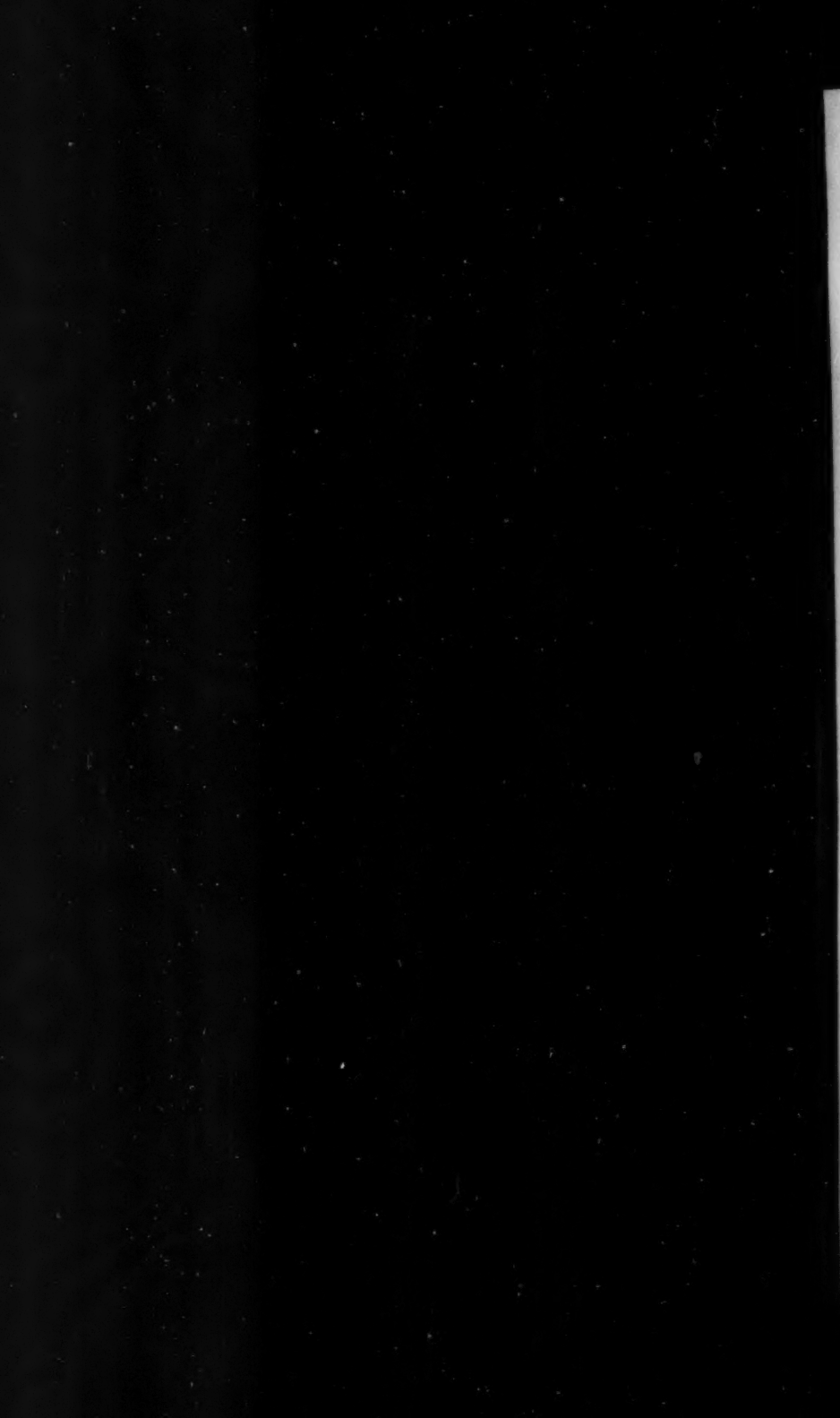
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No. 207.

S. RAMANUJAN.*

By B. M. WILSON, M.A., D.Sc.

GAUSS was the son of a bricklayer, and owed his education to the reigning duke. Ramanujan's father was indeed accountant to a cloth merchant, but it may be doubted whether the position of an Indian accountant in the late nineteenth century was financially very much preferable to that of a German bricklayer in the late eighteenth. At any rate when Ramanujan, a married man of twenty-five, wrote his first letter to Professor Hardy enclosing statements of some of the results he had obtained in mathematics, he himself was employed as a clerk in the Accounts Department of the Port Trust Office of Madras and earning a salary of £20 a year. Ramanujan lived in a more democratic age than Gauss, so there was no reigning duke for him. Fortunate then that there were Indian and British mathematicians who were interested and impressed, and an able and sympathetic civil servant as chairman of the Madras Port authority.

The story of Ramanujan's life has been told in outline many times both in print and from the lecturer's desk. And it is no matter for surprise that it should have been, for it forms an attractive story to tell. A romantic story! Portions of it might be taken over almost unchanged by a scenario-writer for the "talkies"; he need not be a master in his field to have no fear that either the sixpenny or the five-shilling seats would suspect that they were getting anything but what they were accustomed to and knew. The childless marriage of the petty official's daughter, her and her husband's despair, the old father's visit to the temple of Namagiri, his prayer to the goddess that she bless his daughter with children, and (for all these things are related in the biography of Ramanujan that was published after his death) the birth shortly afterwards of the infant Ramanujan. Or his precocity at school: "He used," an old school-fellow writes of him, "to borrow Carr's *Synopsis of Pure Mathematics* from the College library, and delight in verifying some of the formulae given there. He had an extraordinary memory, and could give the values of $\sqrt{2}$, π , e , . . . to any number of decimal places." His school-fellows also report of him that, whatever the lesson, Ramanujan was always working at mathematical problems or performing terrifying arithmetical calculations. This absorption had to be paid for; thus at the age of nineteen he presented himself for the Intermediate Arts Examination of the University of Madras, but failed—presumably on account of his weakness in English, to pass in which subject was (I believe, and probably still is) obligatory. It was at about the time of his failure in this examination that he began to keep his now almost famous note-books.

* An address to the Yorkshire Branch, Feb. 8th, 1930.

In 1909, when he was twenty-one, he married; and permanent employment became necessary. Various Indian mathematicians to whom he had shown the note-books finally secured for him the clerkship under the Madras Port Trust, and at this work he remained, spending all his leisure in filling his note-books, until May 1913, when, by the intervention of several Indian and British civil servants and mathematicians (among them Professor Hardy, to whom Ramanujan had written his first letter in the January of that year), he was given a special scholarship by the University of Madras. From then until his death in 1920 he was able to give his whole time to mathematics.

In view of the events related to have preceded Ramanujan's birth it seems very fitting that the second most important happening in his life, his bringing to England, should also have been made possible only by the intervention of the goddess Namagiri. For Ramanujan's family were Brahmins, and to leave India meant for him loss of caste. Ramanujan's own religious scruples were with difficulty overcome by various Indian mathematicians and by Professor Neville, who was then delivering a course of lectures at the University of Madras. His mother, however, remained obdurate. But, to quote again his Indian biographers, P. V. Seshu Aiyar and R. Ramachandra Rao, "One morning his mother announced that she had had a dream on the previous night, in which she saw her son seated in a big hall amidst a group of Europeans, and that the goddess Namagiri had commanded her not to stand in the way of her son fulfilling his life's purpose. This was a very agreeable surprise to all concerned."

Ramanujan arrived in Cambridge in April 1914, and Professor Hardy was confronted, as he says, with the "great puzzle" of deciding "what was to be done in the way of teaching him modern mathematics." The circumstances were certainly peculiar, and the education of Ramanujan, however delightful and heartening, must indeed have been an exceedingly delicate task. Of it Hardy himself writes, with a modesty which, in view of Ramanujan's subsequent achievements, cannot blind one to the solid reality of his success, "I was afraid that, if I insisted unduly on matters which Ramanujan found irksome, I might destroy his confidence or break the spell of his inspiration. On the other hand there were things of which it was impossible that he should remain in ignorance. . . . It was impossible to allow him to go through life supposing that all the zeros of the Zeta-function were real. So I had to teach him, and in a measure I succeeded, though obviously I learnt from him much more than he learnt from me."

The years in Cambridge did not last long; in the spring of 1917 he fell ill, and in the early summer went into a nursing home. On medical advice he returned to India early in 1919, but died there in April 1920. The last three years of his life were spent very largely in sanatoria and in the sick-bed, and, although some of his most beautiful theorems were discovered at about the time of his election as a Fellow of the Royal Society in 1918, the length of time for which he had both the leisure and the health for intense creative work did not exceed four years. Even if one includes the three years during which he was almost consistently a sick man, Ramanujan's life as a professional mathematician was of only about seven years' duration. His work during that so short time is indeed a remarkable achievement of interest and importance.

In speaking of Ramanujan's mathematical work one may have in mind either the actual results or (so far as one knows them) the methods by which he himself arrived at these results—methods often far different from the final published proofs. The results themselves are often characterised, as to matter, by a certain oddness and unexpectedness, and, as to manner of statement, by an almost startling "snappiness" with which he liked to present them. As Hardy says, "he had a whole library of books by circle-squarers and other cranks"; and that is symptomatic of the trend of his tastes. Or again, "It [Ramanujan's work] has not the simplicity and inevit-

ableness of the very greatest work ; it would be greater if it were less strange " ; but also " One gift it has which no one can deny, profound and invincible originality." Another feature of his work which is to be found illustrated in page after page of his published memoirs and of his unpublished note-books and manuscripts is the amazing insight into algebraical formulæ which he possessed. Where Ramanujan's special gifts and excellence show themselves most clearly is in all that concerns transformations of infinite series and integrals, discovery and establishment of algebraical identities, manipulation of continued fractions, and so forth. " On this side most certainly," Hardy writes in his sketch of the characteristics of Ramanujan's work, " I have never met his equal, and I can compare him only with Euler and Jacobi " ; and indeed if one seeks for a classical work with which to compare in spirit, taste, often even in formal apparatus, much of the best of Ramanujan's writings, it is Jacobi's *Fundamenta Nova* which, almost inevitably, at once occurs to the mind. Yet another side of Ramanujan's mental make-up is illustrated by the unusually frequent and extensive use he made in the discovery of his theorems, especially those in the theory of numbers, of the process of generalisation from particular numerical examples. It was on account of this manner of working that it became true of him that, as Professor Littlewood said, " every positive integer was one of his personal friends " ; so used was he to tabulate and classify the integers from every point of view from which he had occasion to consider them that he knew of each one (beyond limits which most would consider reasonable) wherein it differed from and wherein it resembled other integers. Research into each new problem of the theory of numbers seems almost always to have been preceded by the construction of a table of numerical results, and this was carried usually to a length from which most of us would shrink. And it does not appear that his motives in doing this were utilitarian only ; for he loved arithmetical calculation. The reverse process of specialisation from more general to less general theorems is no less characteristic of his tastes and methods. Thus in his note-books when he has obtained a general formula it is his common practice to set out, as corollaries or examples, a number of particular cases. Yet these particular cases are almost never of the type to be obtained by putting (say) $x=1$ in a known formula ; and the deduction of them from the main theorem may be easy or difficult, but the result of the deduction is always a theorem having points of new and independent interest.

His published papers were republished in one volume by the Cambridge University Press in 1927. Including one posthumous paper which was edited by Hardy from a manuscript of Ramanujan they number thirty-seven. The thirty-six that remain were published during the years 1911-1919 ; of these seven were written in collaboration with Professor Hardy, and twenty-nine by Ramanujan alone, though clearly in preparing for press those papers which were published shortly after his arrival in Cambridge, Ramanujan must have received a good deal of help from Hardy in all that concerns phrasing and manner of presentation.

Ramanujan's earliest papers, those published in 1911-1913, certainly could not be called important. The questions answered are often such as one could quite naturally propose for solution in a paper forming part of an examination for an Honours degree in mathematics—provided always, of course, that one pointed to the method of solution by setting them in the " Hence or otherwise " form. But from the first, though essentially elementary, the work shows some of that " profound and invincible originality " of which Hardy writes.

The shortest but one of the most interesting of these very early papers (its length is only one page) gives a very simple and elegant geometrical construction for the approximate quadrature of the circle, amounting to the construction by ruler and compasses of the length $\sqrt{355/113}$. It is characteristic both of Ramanujan's love of startling statement and of his interest

in the special case that this paper ends with a note that, if the area of the circle is 140,000 square miles, the line obtained by the construction exceeds in length the side of the equivalent square by about an inch. This is equivalent to the statement that, as an approximation to π , $355/113$ is correct to six decimal places; but it is certainly a more sensationally arresting way of making that statement.

Of the papers published by Ramanujan shortly after his arrival in England, one of the most interesting and certainly one of the most clearly characteristic both in the type of problems attacked and in the methods employed in their solution is the very long paper on *Highly Composite Numbers*. Although one of nine papers published during 1915 it extends to over sixty pages, and would have been appreciably longer had not the London Mathematical Society been in financial difficulties and the costs of printing very great at that time. Although it is impossible here to attempt to give the briefest of resumsés of this memoir, it is nevertheless worth while at least to indicate the type of problem with which it deals. Throughout what follows the word "number" means "positive integer." With this understanding a number N is said to be *highly composite* if it is divisible by more factors than all smaller numbers. Thus one easily verifies that the first half-dozen highly composite numbers are (unity being excluded) 2, 4, 6, 12, 24, 36; for instance, 36 has 9 distinct divisors (including 1 and 36), whereas none of the numbers 2, 3, 4, ..., 35 has more than 8. One may say in fact that a highly composite number is, in respect to the number of its factors, a number as unlike a prime as possible. In his paper Ramanujan discusses, first by elementary methods and then by the use of more powerful machinery, the structure of the highly composite numbers; and, quite characteristically, gives a table of the first hundred or so of these numbers. The last entry in this table of the highly composite numbers is

$$N = 6,746,328,388,800 = 2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23.$$

This is in fact the 103rd highly composite number, and it has 10080 divisors. The earliest highly composite number having thirteen digits is the 95th, which has 6912 divisors; it is one-sixth of the number which has just been written out in full.

Ramanujan proves that if a highly composite number N is resolved into its prime factors, and the primes arranged in increasing order of magnitude, so that

$$N = 2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot 7^{a_4} \cdot \dots \cdot p^{a_p},$$

where the indices a_1, a_2, \dots, a_p are positive integers, then necessarily

$$a_1 \geq a_2 \geq a_3 \geq \dots \geq a_p;$$

and he also proves that, except when $N = 4 = 2^2$ or $N = 36 = 2^2 \cdot 3^2$, $a_p = 1$. But for these two facts nothing is known concerning the structure of highly composite numbers which can assist in their calculation; given a number

$$2^{a_1} \cdot 3^{a_2} \cdot 5^{a_3} \cdot \dots \cdot p^{a_p},$$

such that

$$a_1 \geq a_2 \geq a_3 \geq \dots \geq a_p = 1,$$

one has no means of deciding whether or no it is highly composite except by actual trial. One can then imagine what arithmetical labours were involved in the construction of a table extending to the value of N that has been quoted.

The numerical results are sufficient to show that the highly composite numbers are somewhat rare numbers and increase with some rapidity. It is consequently of interest to note that Ramanujan proves that, if N and N'

are two consecutive highly composite numbers ($N' > N$, say), there is a numerical constant k for which

$$N' - N < k \left\{ \frac{N(\log \log \log N)^{\frac{1}{2}}}{\sqrt{(\log \log N)}} \right\}.$$

For from this it follows at once on division by N that, if N is the n th highly composite number, and if $n \rightarrow \infty$, then

$$N'/N \rightarrow 1.$$

It is essential that some mention be made also of the note-books. As has been said, Ramanujan began to keep these at some time round about 1906, and for many years he kept up the habit of jotting down in them results as they arrived to him. The originals are now in the University Library at Madras, but copies have been made and sent to England for the preparation of a proposed annotated edition. The note-books have some five or six hundred manuscript pages, and contain literally thousands of formulae, set down for the most part with no indication of a proof. It is indeed tolerably certain that for an appreciable proportion of the theorems stated in the note-books Ramanujan himself was not in possession of anything that could be recognised as a proof, and that many of them were in consequence really conjectures which he could support by a greater or less body of confirmatory evidence, as, for example, by "proofs" involving invalid passages to the limit, an unjustified use of divergent series, and so forth. Further, statements of theorems made in the note-books almost never include statements of conditions sufficient to ensure the truth of the conclusions; there is of course no reason why they should: the note-books were intended for his own use only, and, at any rate in many cases, sufficient conditions could be readily supplied whenever occasion arose to write out a formal proof. In many cases, but certainly not in all; thus in the earlier pages Ramanujan does in some cases indicate his own proof; and this proof does sometimes involve transformations or assumptions that are never valid—as, for example, the simultaneous assumptions that $|x| < 1$ and that $|x| > 1$! Nevertheless it goes without saying that the note-books do contain a wealth of mathematical theorems both true and interesting; how fascinating can be the verification of those that are true (certainly, when appropriate conditions are added, well over 99 per cent. of the total), and sometimes how even more fascinating the definite disproving of one or two that are demonstrably false, I well know.

In choosing an example to illustrate the Ramanujan of the note-books I intentionally take it from the very early pages; for it seems preferable that some light should now be thrown on the kind of things Ramanujan was occupying himself with in the early twenties, years before he wrote his earliest letter to Professor Hardy. And the example chosen certainly dates from many years before that event. It occurs on page 20 of the manuscript copy of the note-books, and, transcribed verbatim, reads:

"To find convergents to a root of the equation

$$1 = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \dots$$

If

$$P_n = A_1P_{n-1} + A_2P_{n-2} + A_3P_{n-3} + \dots + A_{n-1}P_1,$$

and $P_1 = 1$, then P_n/P_{n+1} approaches x when n becomes greater and greater."

No indication of a proof is given, but there follow six applications of the theorem, four to algebraic equations and two to transcendental equations. Readers may find it of interest to construct a proof of the correctness of Ramanujan's assertion in the following sense:

if the series

$$f(x) = \sum_{n=1}^{\infty} A_n x^n$$

has a radius of convergence greater than $|x_0|$ and if x_0 is the only root of the equation $f(x)=1$ in the region $|x| \leq |x_0|$, then

$$\lim_{n \rightarrow \infty} P_n/P_{n-1} = x_0,$$

where the numbers P_n are defined by Ramanujan's recurrence relation. Thus when the equation is algebraic and has rational coefficients Ramanujan's process defines a sequence of rational approximations to the root of least modulus, assuming such a root to exist. As an application to a transcendental equation Ramanujan puts forward the equation

$$2 = e^x = \sum_{n=0}^{\infty} x^n/n!,$$

so that the root of least modulus is $x = \log_e 2$. The recurrence formula gives

$$P_1 = P_2 = 1, \quad P_3 = \frac{3}{2}, \quad P_4 = \frac{13}{6}, \quad P_5 = \frac{25}{8}, \quad P_6 = \frac{541}{240}, \dots$$

Thus

$$\frac{P_5}{P_6} = \frac{371}{841} = 0.69316 \dots,$$

which is an approximation to the root $\log_e 2$ correct to four decimal figures.

B. M. WILSON.

GLEANINGS FAR AND NEAR.

745. One microscopic glittering point; then another; then another; imperceptible, yet enormous. Yonder light is a focus; that focus is a star; that star is a sun; that sun is a universe; that universe is nothing. For all numbers are as zero in the presence of the Infinite.—Victor Hugo, *The Toilers of the Sea*, ii. i [per Miss Ternouth].

746. For if some such primeval and predestinarian quality were not inherent in the City, how, think you, would the valley of the Serchio . . . lead down pointing straight to Rome; and how would that same line, prolonged across the plain, find, fitting it exactly beyond that plain, this vale of the Elsa, itself leading up directly towards Rome? I say, nowhere else in the world is such a coincidence observable, and they that will not take it for a portent may go back to their rationalism and consort with microbes and make their meals off logarithms, washed down with an exact distillation of the root of minus one; and the peace of fools, that is the deepest and most balmy of all, be theirs for ever and ever.—Belloc, *The Path to Rome*, p. 327 [per Miss Ternouth].

747. *A Catch and its Explanation.*

If	$2 \div 1 = 2.$
And	$2 \div 0 = 2.$
Then	$\frac{2}{1} = 2.$
	$\therefore 1 = 0.$

The fallacy, of course, is that if you divide 2 by 0, the answer is *not* 2 but 0, but it is surprising how many people this simple snare will catch.—*The Royal Magazine*, Feb. 1930 [per Miss W. M. Deans].

748. A mathematician is one who endeavours to secure the greatest possible consistency in his thoughts and statements by guiding the process of his reasoning into those well-worn tracks by which we pass from one relation among quantities to an equivalent relation. He who has kept his mind always in those paths which have never led him or anyone else to an inconsistent result, and has traversed them so often that the act of passage has become rather automatic than voluntary, is, and knows himself to be, an accomplished mathematician.—*Nature*, Quaternions, p. 137, vol. ix.

SOME INEQUALITIES CONNECTED WITH A METHOD OF REPRESENTING POSITIVE INTEGERS.

BY F. S. MACAULAY, F.R.S.

In ordinary arithmetic a number $ab \dots jk$ represents the sum of the place values of its digits a, b, \dots, j, k , these place values being $k, 10j, 10^2i$, etc. Also the representation is unique if the digits are restricted to being positive and less than 10. We are going to consider the representation of whole numbers by means of digits with place values, but with a different definition of place value. The digits $a, b, \dots, j, k+1$ of a given number A are to be positive, and the number of them, n , is to be assigned, the units digit $k+1$ (in the standard form) is to be greater than zero, and the place value of the digit a in the n th place counting from $k+1$ is to be $a \cdot a+1 \dots a+n-1$ divided by $1 \cdot 2 \dots n$, or the number of homogeneous products of a letters taken n at a time; this number or place value is written $(a)_n$. Hence

$$A = ab \dots jk+1 = (a, b, \dots, j, k+1)_n = (a)_n + (b)_{n-1} + \dots + (j)_2 + (k+1)_1.$$

Finally we say that A is in its *standard form* if its digits are subjected to the conditions $a \geq b \geq \dots \geq j \geq k \geq 0$. We first show that this standard form for A is unique.

The homogeneous products of $a+1$ letters x_0, x_1, \dots, x_a taken n at a time consist of all those including x_0 , of which there are $(a+1)_{n-1}$, and all those not including x_0 , of which there are $(a)_n$. Hence

$$(a+1)_n = (a+1)_{n-1} + (a)_n = (a, a+1)_n = (a, a, a+1)_n = \text{etc.},$$

where the suffix n indicates the number of digits, including zero digits.

The first digit a (or the digit in the n th place) is the unique number a such that $(a)_n < A \leq (a+1)_n$; and the remainder B , on taking $(a)_n$ from A , is such that $0 < B \leq (a+1)_n - (a)_n \leq (a+1)_{n-1}$. The second digit b is the unique number b such that $(b)_{n-1} < B \leq (b+1)_{n-1}$; and since $B \leq (a+1)_{n-1}$, we have $(b)_{n-1} < (a+1)_{n-1}$, or $b \leq a$. Similarly all the digits up to the last but one j are uniquely determined. If $B=1$, all the digits from b to j are zeros. The last digit $k+1$ must be K , the remainder in the previous step $(j)_2 < J \leq (j+1)_2$ on taking $(j)_2$ from J , viz. $0 < K \leq (j+1)_1$. Hence $0 < k+1 \leq j+1$, or $0 \leq k \leq j$. Hence any given number A can be represented uniquely with n digits in the form $A = (a, b, \dots, j, k+1)_n$ where

$$a \geq b \geq \dots \geq j \geq k \geq 0.$$

Several details should be noticed. The last (or units) digit $k+1$ is the only one allowed to exceed the previous digit j , and that by not more than 1. When this happens, i.e. when $k+1 > j$, A can be expressed in a second unique form, different from the standard form, namely, if the p th digit of A is the first which equals k , the *second form* is $A = (a, b, \dots, k+1)_n$ with p digits > 0 , and one or more digits equal to 0, which are generally not written; so that the last digit $k+1$ is the same in both forms, and in the second form no digit is allowed to exceed the previous one. If the last digit $k+1$ is 1, A is named *special** or a *special number*. (If A has a second form, $A+1$ is special, and vice versa.) The advantage of the standard form is that properties and rules referred to it have less exceptions than when they are referred to the second form. In both forms the place value of every digit 1 is 1, and of every digit 0 is 0.

Definition. QA , or $Q_n A$, denotes the number obtained by adding 1 to each of the digits of A when expressed in the standard form, zero digits being excepted when A is expressed in the second form. Similarly $Q^{-1}A$, or $Q_n^{-1}A$, denotes the number obtained by subtracting 1 from each of the digits of A ,

* The number of special numbers $\leq (a)_n$ is $(a)_{n-1}$, and of non-special $(a)_n - (a)_{n-1}$ or $(a-1)_n$; and the ratio is that of n to $a-1$.

the zero digits excepted, negative digits being inadmissible. This rule applies to both forms. $Q^p A$ means that all the digits of A , when in standard form, are to be increased by p , and $Q^{-p} A$ that all digits are to be diminished by p , except digits $< p$, which become zeros in $Q^{-p} A$.

To see how QA and $Q^{-1}A$ change as A increases by steps of 1, it is easiest to imagine A expressed in the second form, when it has a second form. Then when A has an increase 1, the 1 takes the place of the first zero digit. In QA this 1 becomes 2, and so QA has an increase of $(2)_r$, where r is the number of zero digits in A ($r \geq 1$); while $Q^{-1}A$ remains stationary. Thus QA has an increase ≥ 2 , and $Q^{-1}A$ an increase 0, at each special value of A ; and similarly both QA and $Q^{-1}A$ have an increase 1 at each non-special value of A . The maximum increase for QA is n , viz. at $A = (a, 1)_n$; or $n+1$ at $A=1$. The increase of QA at the value $A=K$ means $QK - Q(K-1)$.

When A is non-special, $QQ^{-1}A = A$; but when A is special, $QQ^{-1}A = A - q$, where q is the number of unit digits in A . Thus $QQ^{-1}A \leq A$; but $Q^{-1}QA = A$ always, because QA is non-special.

$$\begin{aligned} \text{If } A &= (a_1, a_2, \dots, a_{n+1})_n, \\ \text{then } A - Q^{-1}A &= (a_1, a_2, \dots, a_{n+1})_n - (a_1 - 1, a_2 - 1, \dots, a_n)_n \\ &= (a_1, a_2, \dots, a_{n-1} + 1)_n, \dots \dots \dots (1) \end{aligned}$$

which is the general number in standard form in $n-1$ digits; and

$$\begin{aligned} Q_{n-1}(A - Q^{-1}A) &= (a_1 + 1, a_2 + 1, \dots, a_{n-1} + 2) \\ &= QA - A. \dots \dots \dots (2) \end{aligned}$$

This relation is rather important. Similarly $Q_{n-1}(QA - A) = Q^2A - QA$, etc. Any given number C can be expressed uniquely as $C_1 - Q^{-1}C_1$, where C_1 is special; viz. if $C = (c_1, c_2, \dots, c_{n-1} + 1)_{n-1}$, then $C_1 = (c_1, c_2, \dots, c_{n-1}, 1)_n$. Also, if $c_{n-1} \neq 0$, then $C_1 + 1, \dots, C_1 + c_{n-1}$ are non-special solutions of the equation $X - Q^{-1}X = C$, where C_1 is the single special solution.

The first and principal theorem to be mentioned is:

I. If $Q^{-1}A \leq B \leq A$, then $QA - QB \leq Q_{n-1}(A - B)$, excluding special values of A for which $Q^{-1}A = B$.

It is evident the theorem holds for $B=A$; and it will also hold for $B=Q^{-1}A$ provided $QA - QQ^{-1}A \leq Q_{n-1}(A - Q^{-1}A) \leq QA - A$, by (2) above. This does hold when A is non-special ($QQ^{-1}A = A$), but not when A is special ($QQ^{-1}A < A$). As regards the remaining values of B , viz. $Q^{-1}A < B < A$, the theorem is proved by E. Sperner, in the *Abhandlungen aus dem Math. Seminar der Hamburgischen Univ.* vii. Heft 2/3, 149-163 (1929), with differently expressed conditions. He proves that if $A - B = C > 0$; and $A^{(n)} > A + C^{(n-1)}$, or (in our notation) $Q_{n-1}A > A + Q_{n-2}C$; then $A^{(n)} \leq B^{(n)} + C^{(n-1)}$, or $Q_{n-1}A - Q_{n-1}B \leq Q_{n-2}C$. Thus his data are $C > 0$, or $A > B$; and $Q_{n-2}C < Q_{n-1}A - A < Q_{n-2}(A - Q_{n-1}A)$, or $C < A - Q_{n-1}A$, or $B > Q_{n-1}A$; i.e. his data are $Q_{n-1}A < B < A$, and the conclusion

$$Q_{n-1}A - Q_{n-1}B \leq Q_{n-2}(A - B);$$

which is the same as the statement above, with n changed to $n-1$. We shall not give the proof of the theorem but consider its consequences only.

The data of I. viz. $Q^{-1}A \leq B \leq A$, excluding special values of A for which $Q^{-1}A = B$, are exactly equivalent to the simpler expressed data $B \leq A \leq QB$. For $Q^{-1}A < B$ is equivalent to $A < QB$, since $Q^{-1}A$ has an increase at $A=QB$ and is then equal to B ; and $Q^{-1}A = B$ is equivalent to $QQ^{-1}A = QB$, or $A - q = QB$, where the values of A for which $q \neq 0$ are special and excluded; so that $Q^{-1}A = B$ with the exclusion datum is equivalent to $A = QB$. Hence the two sets of data are equivalent, the datum $B \leq A$ being common to both sets. Hence Theorem I. may also be expressed.

If $B \leq A \leq QB$, then $QA - QB \leq Q_{n-1}(A - B)$, or $QA - QB \leq QC_1 - C_1$.

In this statement of I. there is still a (tacit) restriction, namely, that the third term QB in the data $B \leq A \leq QB$ is non-special. It suggests, however, a third statement from which the restriction is removed.

II. If $Q^{-1}A \leq B \leq A$, then $QB - A \leq Q_{n-1}(B - Q^{-1}A)$.

If, in this statement (or new theorem), A is non-special, it can be replaced by QA' , and the theorem becomes: If $A' \leq B \leq QA'$, then

$$QB - QA' \leq Q_{n-1}(B - A'),$$

which is the second statement of I.

To prove II. From $B \leq A$ follows $Q^{-1}B \leq Q^{-1}A$, and from $Q^{-1}A \leq B \leq A$ follows $Q^{-1}B \leq Q^{-1}A \leq B$. Applying I. to these, we have

$$QB - QQ^{-1}A \leq Q_{n-1}(B - Q^{-1}A),$$

excluding the case of $Q^{-1}B = Q^{-1}A$. But $QQ^{-1}A \leq A$; hence, by adding, $QB - A \leq Q_{n-1}(B - Q^{-1}A)$, when $Q^{-1}B \neq Q^{-1}A$. Also, when $Q^{-1}B = Q^{-1}A$, we have $Q_{n-1}(B - Q^{-1}A) = Q_{n-1}(B - Q^{-1}B) = QB - B = QB - A + (A - B) \geq QB - A$. Hence $QB - A \leq Q_{n-1}(B - Q^{-1}A)$ in all cases, Q.E.D.

By applying II. to $Q^{-1}B \leq Q^{-1}A \leq B$ (proved six lines above), we have

$$QQ^{-1}A - B \leq Q_{n-1}(Q^{-1}A - Q^{-1}B);$$

and similar results from $Q^{-2}A \leq Q^{-1}B \leq Q^{-1}A$, etc.

Also, if A is non-special, from $Q^{-1}A \leq B$ follows $QQ^{-1}A \leq QB$, or $A \leq QB$, and $A \leq QB \leq QA \leq Q^2B \leq Q^3A \leq \dots$, from which, by II. it follows that

$$Q^2B - QA \leq Q_{n-1}(QB - A) \leq Q_{n-1}^2(B - Q^{-1}A),$$

$$Q^3A - Q^2B \leq Q_{n-1}(QA - QB) \leq Q_{n-1}^2(A - B),$$

and similar results.

For the rest of what follows we shall write:

$$A = (a_1, a_2, \dots, a_n + 1)_n, \quad B = (b_1, b_2, \dots, b_n + 1)_n,$$

$$A - B = C = (c_1, c_2, \dots, c_{n-1} + 1)_{n-1} = C_1 - Q^{-1}C_1, \quad C_1 = (c_1, c_2, \dots, c_{n-1}, 1)_n.$$

III. $A - B = C$; put $B - Q^{-1}A = D$; then the data of both I. and II. are fulfilled by $C \geq 0$, $D \geq 0$. By adding, $C + D = A - Q^{-1}A$, and

$$Q_{n-1}(C + D) = Q_{n-1}(A - Q^{-1}A) = QA - A = (QA - QB) + (QB - A) \\ \leq Q_{n-1}C + Q_{n-1}D, \text{ by I. and II.,}$$

or $QC + QD \geq Q(C + D)$, when C, D are any positive integers.

This may also be written $QC \geq Q(C + D) - QD$, or (writing C for $C + D$)

$$Q(C - D) \geq QC - QD, \quad C \geq D.$$

Keeping $C - D$ fixed we can choose C and D so that $C = (d, d_1, d_2, \dots)_n$, $D = (d)_n$, in which case $QC - QD = Q_{n-1}(C - D)$. Hence

$$Q(C - D) \geq Q_{n-1}(C - D), \text{ or } Q_n C \geq Q_{n-1} C.$$

IV. Between the limits $B \leq A \leq QB$ for A choose narrower limits

$$(b_1 + 1)_n \leq A \leq (b_1 + 1, b_2, \dots, b_{n-1}, b_n + 1)_n.$$

$$\text{Put } B = (b_1, b_2, \dots, b_n + 1)_n = (b_1)_n + B', \quad B' \leq (b_1 + 1)_{n-1},$$

$$A = (b_1 + 1)_n + A', \quad 0 \leq A' \leq B'.$$

Then A', B' are any two numbers such that $0 \leq A' \leq B' \leq (b_1 + 1)_{n-1}$.

$$\text{Now } C = A - B = (b_1 + 1)_{n-1} + A' - B',$$

$$\text{i.e. } (b_1 + 1)_{n-1} - C = B' - A' \geq 0.$$

$$\text{And } QA - QB \leq Q_{n-1}C,$$

$$\text{i.e. } Q(b_1 + 1)_n + Q_{n-1}A' - Q(b_1)_n - Q_{n-1}B' \leq Q_{n-1}C,$$

$$\text{or } Q_{n-1}(b_1 + 1)_{n-1} - Q_{n-1}C \leq Q_{n-1}B' - Q_{n-1}A',$$

$$\text{where } (b_1 + 1)_{n-1} - C = B' - A' \geq 0, \text{ as above.}$$

Expressed in words (changing $n-1$ to n) this is: For a constant interval $B'-A'$ the value of $QB'-QA'$ is as small when $B'=(b_1+1)_n$ as when B' has any less value; or if $B'-A'$ is constant, and $B' \leq (b_1+1)_n$, the value of $QB'-QA'$ is a minimum at $B'=(b_1+1)_n$.

V. Let C , or $A-B$, be constant. Then we have a constant interval $A-B$ which may be said to be higher or lower in the number scale according as A (and B) is higher or lower. The data, or conditions, for $QA-QB \leq Q_{n-1}C$, are $Q^{-1}A \leq B \leq A$, with $Q^{-1}A < B$ when A is special. These are satisfied by $Q^{-1}A \leq A-C \leq A$, with $Q^{-1}A < A-C$ when A is special; or $C \geq 0$, and $A-Q^{-1}A \geq C$, with $A-Q^{-1}A > C$ when A is special. The value $A=C_1$ is inadmissible, because then $A-Q^{-1}A=C$ and A is special. But all values of $A > C_1$ satisfy the conditions. For, if A is an increasing number, $A-Q^{-1}A$ increases in exactly the opposite way to that in which $Q^{-1}A$ increases. At any non-special value of A , $Q^{-1}A$ has an increase of 1 and $A-Q^{-1}A$ is stationary; and at any special value of A , $Q^{-1}A$ is stationary and $A-Q^{-1}A$ has an increase of 1. Thus, if at $A=1+C_1$, A is special ($c_{n-1}=0$), then $A-Q^{-1}A > C$, and A is admissible; and if at $A=1+C_1$, A is non-special, $A-Q^{-1}A=C$, and A is still admissible; and similarly for all subsequent values of A . Hence we have the following theorem, which is Theorem I. in a different guise.

For a given interval $A-B=C=C_1-Q^{-1}C_1$ the value of $QA-QB$ never exceeds $Q_{n-1}C$, or QC_1-C_1 , when $A > C_1$; and $QA-QB$ has this value at $A=1+C_1$, $B=1+Q^{-1}C_1$, and also at any value $A=(b)_n+C > C_1$. When $A=C_1$, $QA-QB > Q_{n-1}C$.

The smaller the interval $A-B=C$ the lower in the number scale is the position $A=C_1$, $B=Q^{-1}C_1$ at which $QA-QB > Q_{n-1}C$, and after which $QA-QB \leq Q_{n-1}C$. The absolutely highest value of $QA-QB$ is QC at $A=C$, and the absolutely lowest value is C at $A=(C+1)_n$.

VI. Take a number N such that $(b)_n \leq N \leq (b+1)_n$. Then

$$QN - Q(b)_n = Q_{n-1}\{N - (b)_n\};$$

and, applying V. to an interval $N - (b)_n \leq (b+1)_{n-1}$,

$$Q(N+C) - Q\{(b)_n+C\} \leq Q_{n-1}\{N - (b)_n\} \leq QN - Q(b)_n,$$

or

$$Q\{(b)_n+C\} - Q(b)_n \geq Q(N+C) - QN,$$

where the interval is now C . This is true for any $C > 0$, and any N such that $(b)_n \leq N \leq (b+1)_n$. Hence, putting $N=(b+1)_n$,

$$Q\{(b)_n+C\} - Q(b)_n \geq Q\{(b+1)_n+C\} - Q(b+1)_n \geq Q(N_1+C) - QN_1,$$

where $(b+1)_n \leq N_1 \leq (b+2)_n$, and so on. Thus we have

$$Q\{(b)_n+C\} - Q(b)_n \geq Q(N+C) - QN$$

so long as $C > 0$, and $N \geq (b)_n$. This is the complement of IV.

For any constant interval $A-B$ which moves up the number scale the value of $QA-QB$ at $B=(b)_n$ is as high as for any higher place of the interval; and at $A=(a)_n$ is as low as for any lower place of the interval (IV.).

F. S. M.

749. Laplace . . . proposed to supersede the Christian era by an astronomical one, which might be recognized in all countries, and fixed it to A.D. 1250, when the lines of the equinoxes and apsides were at right angles.—*Athenaeum*, 1844, p. 924.

750. "The two extreme parts of the axis called the north and south poles are according to the present system of astronomy of equal form, although that form is of an oblate or flat description." We suspect our author himself to be more than usually flat in the poll.—H. C. Johnson, *The Invisible Universe Disclosed* [*Athenaeum*, 1844, p. 770].

THE PARTICULAR INTEGRALS OF A CLASS OF LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS, WITH SPECIAL REFERENCE TO A FORMULA GIVEN BY FORSYTH.*

BY F. UNDERWOOD, M.Sc.

1. The linear differential equation with constant coefficients may be written

$$f(D)y = V, \dots\dots\dots(1)$$

where $f(D)$ is a polynomial in $D \left(= \frac{d}{dx} \right)$ with constant coefficients and V is a function of x . If V contains an integral power of x as a factor, we may write $V = x^m T$, and for the determination of the particular integral use Forsyth's formula * (really an extended form of Leibnitz's theorem),

$$y = \frac{1}{f(D)} x^m T \\ = x^m \frac{1}{f(D)} \cdot T + mx^{m-1} \left\{ \frac{d}{dD} \frac{1}{f(D)} \right\} T + \frac{m(m-1)}{2!} x^{m-2} \left\{ \frac{d^2}{dD^2} \frac{1}{f(D)} \right\} T \\ + \dots + \left\{ \frac{d^m}{dD^m} \frac{1}{f(D)} \right\} \cdot T. \dots\dots\dots(2)$$

It seems to be essential, though probably not generally known, that, in the application of this formula, the successive operators

$$\frac{1}{f(D)}, \left\{ \frac{d}{dD} \frac{1}{f(D)} \right\}, \dots \left\{ \frac{d^m}{dD^m} \frac{1}{f(D)} \right\}$$

must be expressed in partial fractions before being used to operate on the function T . It will be shown below in a case which is fairly general, and in some easy examples, that other common (and apparently sound) methods lead to serious omissions in the particular integral which is required in (2).

The development of this formula may be illustrated by the case when $f(D) = (D - a_1)(D - a_2) \dots (D - a_n)$ and $T = e^{a_1 x}$. An easier case is given when $T = e^{kx}$, where k is not equal to any of the a 's, and a more complicated case arises when $f(D)$ contains repeated factors $(D - a_1)$, but that considered is sufficient for the present purpose.

Since
$$\frac{1}{f(D)} = \sum_{r=1}^n \frac{1}{f'(a_r)} \cdot \frac{1}{D - a_r},$$

$$\frac{d^s}{dD^s} \left\{ \frac{1}{f(D)} \right\} = (-1)^s \cdot s! \sum_{r=1}^n \frac{1}{f'(a_r)} \cdot \frac{1}{(D - a_r)^{s+1}}.$$

Also
$$\frac{1}{(D - a_r)^p} \cdot e^{a_1 x} = \frac{e^{a_1 x}}{(a_1 - a_r)^p}, \quad (a_r \neq a_1),$$

but
$$\frac{1}{(D - a_1)^p} \cdot e^{a_1 x} = e^{a_1 x} \cdot \frac{1}{D^p} \cdot 1 = e^{a_1 x} \cdot \frac{x^p}{p!}.$$

Hence (2) gives the particular integral

$$y = \sum_{s=0}^m {}^m C_s (-1)^s s! x^{m-s} \left[\sum_{r=1}^n \frac{1}{f'(a_r)} \cdot \frac{1}{(D - a_r)^{s+1}} \right] e^{a_1 x} \\ = e^{a_1 x} \sum_{s=0}^m \frac{(-1)^s m! x^{m-s}}{(m-s)!} \left[\frac{1}{f'(a_1) (s+1)!} + \sum_{r=2}^n \frac{1}{f'(a_r)} \cdot \frac{1}{(a_1 - a_r)^{s+1}} \right] \dots (A)$$

* *Treatise on Differential Equations*, p. 78, iv.

In this result the coefficient of $x^{m+1}e^{a_1x}$ is

$$\begin{aligned} & \sum_{s=0}^m \frac{(-1)^s m!}{(m-s)!(s+1)!} \cdot \frac{1}{f'(a_1)} \\ &= \frac{1}{f'(a_1)} \left[t^{-m} C_1 \frac{t^2}{2} + m C_2 \frac{t^3}{3} - \dots + (-1)^s m C_s \frac{t^{s+1}}{s+1} + \dots \right]_{t=1} \\ &= \frac{1}{f'(a_1)} \int_0^1 (1-t)^m dt = \frac{1}{f'(a_1)} \cdot \frac{1}{m+1}. \end{aligned}$$

It may also be noted that the last term in (A), viz.

$$e^{a_1x} (-1)^m \cdot m! \sum_{r=2}^n \frac{1}{f'(a_r)} \frac{1}{(a_1 - a_r)^{m+1}},$$

may be included in the complementary function, but every other term in (A) must be retained as part of the particular integral.

Also it is quite easy to verify that the coefficient of every term of the type $x^s e^{a_1x}$ agrees with that obtained by the method usually applied for this case, viz.

$$\frac{1}{f(D)} (x^m e^{a_1x}) = e^{a_1x} \frac{1}{f(D+a_1)} \cdot x^m. \dots\dots\dots (B)$$

2. In the development of Forsyth's formula leading to result (A), it will be noted that each operator was expressed as a sum of operators, i.e. in partial fractions, before it was used to operate on the exponential term e^{a_1x} . If this is not done, and, in particular, if any operator is expressed in factors and used in such a way as to lead to an arbitrary selection of order of its factors, errors will arise, even though the method used is an ordinary one that is applied successfully to evaluate simple cases of particular integrals. The errors referred to usually consist of the omission of some terms of the result (A).

Thus, taking the first term $x^m \cdot \frac{1}{f(D)} \cdot T$ in the formula (2), a method commonly used to evaluate $\frac{1}{f(D)} \cdot T$ in the case considered is :

$$\begin{aligned} \frac{1}{f(D)} e^{a_1x} &= \frac{1}{(D-a_1)(D-a_2)\dots(D-a_n)} \cdot e^{a_1x} \\ &= \frac{1}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)} \frac{1}{D-a_1} e^{a_1x} \\ &= \frac{1}{f'(a_1)} \cdot e^{a_1x} \cdot \frac{1}{D} \cdot 1 = \frac{x e^{a_1x}}{f'(a_1)}. \end{aligned}$$

When multiplied by x^m this gives the leading term $\frac{x^{m+1}e^{a_1x}}{f'(a_1)}$ which must arise from the development of $x^m \cdot \frac{1}{f(D)} \cdot T$ in (2), but it omits all the other terms $x^m e^{a_1x} \sum_{r=2}^n \frac{1}{f'(a_r)} \cdot \frac{1}{a_1 - a_r}$ which should arise from it.

If the second operator $\frac{d}{dD} \frac{1}{f(D)}$ were treated in the same way we should obtain the correct coefficient of $x^{m+1}e^{a_1x}$, but all the terms containing $x^{m-1}e^{a_1x}$ would be omitted ; and so for the later operators.

It will be noted that the result obtained immediately above in using the first operator, $\frac{x e^{a_1x}}{f'(a_1)}$, would be correct for the particular integral if $\frac{1}{f(D)} e^{a_1x}$

alone had to be evaluated, but the presence of the factor x^m renders necessary the strict determination by partial fractions, as in result (A).

This may be illustrated by means of a few very simple forms of $f(D)$, such as those in the following examples:

Example 1. $f(D) = D^2 - 1$; $a_1 = 1$.

$$\frac{1}{f(D)} \cdot T = \frac{1}{D^2 - 1} \cdot e^x = \frac{1}{2} \left(\frac{1}{D-1} - \frac{1}{D+1} \right) e^x \\ = \frac{1}{2} e^x (x - \frac{1}{2}).$$

The usual way of calculating this for a particular integral, viz.

$$\frac{1}{D^2 - 1} e^x = \frac{1}{(D-1)(D+1)} e^x = \frac{1}{2} \frac{1}{D-1} e^x = \frac{e^x}{2} \cdot \frac{1}{D} \cdot 1 = \frac{x}{2} e^x,$$

is not sufficient here, for in the application of Forsyth's formula, the part $-\frac{1}{2}e^x$ cannot be omitted if $m > 0$, because when multiplied by x^m it forms an essential part of the particular integral.

Similarly, the second operator $\frac{d}{dD} \left\{ \frac{1}{f(D)} \right\} = -\frac{f'(D)}{\{f(D)\}^2} = \frac{-2D}{(D^2-1)^2}$ must be taken in the form $-\frac{1}{2} \left\{ \frac{1}{(D-1)^2} - \frac{1}{(D+1)^2} \right\}$ in order that no part of the P.I. may be omitted in applying the formula. Any order of operations that may be indicated by the symbols $-\frac{1}{\{f(D)\}^2} f'(D)$, or $-f'(D) \cdot \frac{1}{\{f(D)\}^2}$, or even $-\frac{1}{f(D)} f'(D) \cdot \frac{1}{f(D)}$ must omit some part of the P.I. when the final result of these operations is multiplied by x^{m-1} , except, of course, in the case $m=1$.

The same considerations apply to all the succeeding operators, which may be called the higher differential coefficients of $\frac{1}{f(D)}$. In every case the partial fractions form is necessary so long as the result of the operations is multiplied by an integral power of x .

Example 2.* $(D^2 + 1)y = x \sin x$.

Using $I(\xi) = I(\xi + i\eta) = \eta$, as usual, the two terms derived from formula (2) may be obtained as follows:

$$x \cdot \frac{1}{f(D)} \cdot T = x \cdot I \left[\frac{1}{D^2 + 1} \cdot e^{ix} \right] = x \cdot I \left[\frac{1}{2i} \left(\frac{1}{D-i} - \frac{1}{D+i} \right) e^{ix} \right] \\ = x \cdot I \left[\frac{e^{ix}}{2i} \left(x - \frac{1}{2i} \right) \right] = x \left(-\frac{x}{2} \cos x + \frac{1}{4} \sin x \right) \\ = -\frac{x^2}{2} \cos x + \frac{x}{4} \sin x.$$

As in Example 1, it will be noted that the term $\frac{x}{4} \sin x$ would be missed by a method of operating which did not put $\frac{1}{f(D)}$ into partial fractions.

$$\frac{-f'(D)}{\{f(D)\}^2} \cdot T = \frac{-2D}{(D^2 + 1)^2} \cdot \sin x = I \left[\frac{i}{2} \left\{ \frac{1}{(D-i)^2} - \frac{1}{(D+i)^2} \right\} e^{ix} \right] \\ = I \left[\frac{ie^{ix}}{2} \left(\frac{x^2}{2} + \frac{1}{4} \right) \right] \\ = \frac{x^3}{4} \cos x + \frac{1}{8} \cos x.$$

* This example was sent to Prof. Piaggio by Mr. E. J. A. Barnard of the University of Melbourne in connection with the formula (2), and the present note had its origin here.

The part $\frac{1}{4} \cos x$ is what was called the last term in result (A) and may be included in the complementary function.

Thus the particular integral is $\frac{1}{4} (-x^2 \cos x + x \sin x)$.

Example 3. $(D-1)(D-2)(D-4)y = x^m e^{ax}$, where m may be any positive integer.

3. It is obvious that in these examples the ordinary method of calculating the P.I. {as in (B) above} is more convenient than the formula (2) and its development leading to result (A), but the latter is not without interest in itself, and the point discussed above concerning the necessity of the partial fractions form of the operators is one which might easily be overlooked. The same form is necessary when T in $V = x^m T$ takes other forms instead of exponentials such as e^{ax} . One easy case of this ($m=1$, $T = \log x$) is given in Forsyth's *Treatise on Differential Equations*, p. 80, Ex. 18. In the general case when partial fractions are used, all the operators are of the type $\frac{1}{(D-a)^s}$, and

$$\frac{1}{(D-a)^s} \cdot T = e^{ax} \left\{ \int \int \dots \int T e^{-ax} (dx)^s \right\}.$$

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751. "The work [*A Treatise on the Conic Sections in Five Books*, 1794, by G. Walker, F.R.S.] which is here offered to the public is the fruit of many years agreeable attention, to which the Author was led by a persuasion that so simple a property as is exhibited in the 24 Prop. of the Universal Arithmetic, might contain the elements of all the properties of the three Conic Sections. . . . It is now near thirty years since he first discovered the property of the Generating Circle, but though it be an immediate consequence of the primary propositions, yet it was for some time hid from his view, nor does it appear that any Geometer had discovered it, though the property in Sir Isaac Newton's Arithmetic has so long been known."—[From the Preface, p. v. *loc. cit.* Cf. C. Taylor, *Ancient and Modern Geometry of Conics*, 1881, Sec. III. p. lxxii.]

752. After referring to the actual loss that he fears he must sustain by the publication of the work (pp. 218 + 207 beautiful copperplates) he proceeds :

Perhaps it was the misfortune of the Author to be a Dissenter, when it has become the temper and the very principle of the day to cut off a Dissenter from every public expectation. But surely, however wise the general interdict may be, pure innocent science might have promised itself an exemption from the malediction both of political and religious party.—[Pref. *loc. cit.* p. x.]

753. His [Newton's] views were too contracted for astronomy. He could not enter into the majesty of the celestial movements. . . . One might as well look at Jupiter through a microscope as study astronomy under Sir Isaac Newton's auspices. The universal Creator framed Sir Isaac to analyze aliquid quantities, not to designate immensities.—*Elective Polarity the Universal Agent*, by Frances Barbara Burton [*Athenaeum*, 1846, p. 434].

754. I know not whether it is wise to apply your mind to geometry, though it is a noble study and well worthy of a fine understanding, but . . . you are not over cheerful by nature, and it is a study which will make you still more grave ; and as it requires the strongest application of the mind, it is likely to wear out the powers of the intellect, and very much to impair the health . . . and you know you have no health to spare.—Hubert Languet to Sir Philip Sidney [*Athenaeum*, 1846, p. 1238].

755. On Zach's statement that Delambre stole some of his formulae, Arago exclaimed : "pourquoi le ferait-il ? Dieu sait qu'il n'en a que trop."

MULTIPLICATION AND DIVISION OF DECIMALS.*

R. A. M. KEARNEY, B.A.

DISPENSING WITH STANDARD FORM BY THE USE OF A NEW (OR POSSIBLY VERY OLD) METHOD OF ARRANGEMENT.

It is perhaps somewhat remarkable that in such frequently recurring and comparatively simple processes as those of multiplication and division of decimals there should still be no final settling down by teachers of Arithmetic into one standardized method and arrangement of work, shown by experience to be the simplest, safest, least laborious, and most easily understood and assimilated by the pupil. Indeed there seems a tendency in some quarters, after trying in succession all the various modifications of method and set-out which have been suggested, each as an improvement on its predecessor, to come round full circle to the older methods in use, say, fifty years ago.

The problem is, of course, complicated by the introduction, at the later stages of the pupil's work, of contracted methods, which render inoperative the simple method of working with whole numbers and afterwards "counting the decimal places." Under the pressure of this fact there is even a disposition to abandon the use of contracted methods altogether. Those who appreciate both the extensive practical application of these methods and also the intrinsic beauty of an arrangement of the work which preserves only the figures which are useful in arriving at the desired result, will certainly regret what to them seems a retrogressive tendency.

It is my purpose here to describe a method of set-out of multiplication and division which has the following advantages:

It can be applied without a break from the early stages in the teaching of Arithmetic.

The arrangement is the same for whole numbers and for decimals, and whether contracted methods are used or not.

The position of the decimal point is fixed automatically by rule.

Standard form is dispensed with.

The rules for contraction are simple and obvious, once they are pointed out.

Division is exhibited plainly as the inverse of multiplication.

Before describing the method referred to I would like first to make a few remarks regarding the use of standard form.

It is certainly desirable that all students should at some stage in their study become familiar with the expression of numbers in "standard form." If not done previously, it should be introduced as a preliminary to the study of logarithms. This will enable the pupil to understand without difficulty the sometimes puzzling subject of "characteristics."

Thus $92,800,000 = 9.28 \times 10^7$

and $0.000572 = 5.72 \times 10^{-4}$,

whence it is obvious that the characteristics of the logarithms of these numbers are respectively 7 and -4.

The student will also be prepared for the method universally adopted in Physics for expressing large and small numbers.

Where standard form is used for the multiplication and division of decimals, the pupil often experiences difficulty owing to the necessity of "shifting the decimal point" in opposite directions in the multiplier and multiplicand, and in the same direction in the divisor and dividend. This difficulty may be lessened by the following arrangement of the preliminary work:

$$(a) \begin{array}{r} 7.8264 \times 3.596 \\ 78264 \times 3596 \end{array}$$

$$(b) \begin{array}{r} 853.1 \times 0.00543 \\ = 0.8531 \times 5.43 \end{array}$$

$$(c) \begin{array}{r} 1234.5 \\ 69.23 \end{array} = \begin{array}{r} 12345 \\ 6923 \end{array}$$

* A paper read to the London Branch of the Mathematical Association on March 16th, 1929.

In multiplication the figures in the row following the equal sign are written exactly under the same figures in the row preceding the equal sign. The decimal point in the second factor is then inserted in the desired position and, if this position is, for instance, two places nearer to the multiplication sign than the decimal point in the number immediately above, then the decimal point in the first factor is also written two places nearer to the multiplication sign than the decimal point in the number above it.

In division the sign \div is avoided altogether and the fractional form used instead. The divisor (that is, the denominator) is placed with its decimal point and figures respectively under the decimal point and corresponding figures of the dividend (or numerator). The fraction after the equal sign is written with the figures in exactly the same relative positions as in the preceding fraction; the decimal point in the denominator is then inserted in the position desired and the decimal point in the numerator is placed exactly above it.

There need be no mention of "shifting the decimal point" (if this is deemed confusing to the pupil). We are simply writing down another pair of numbers whose product, or quotient, is exactly the same as the product, or quotient of the first pair, and once the pupil sees clearly that the rule given above does yield an equivalent pair of numbers (as regards their product, or quotient) he can in future simply follow the rule mechanically.

The symmetrical arrangement of the four decimal points should be noted. The axis of symmetry is in one case vertical (roughly the line drawn through the two multiplication signs) and in the other case horizontal (the line separating the numerator and denominator of the fraction). Thus:

$$= \begin{array}{c} \cdot \\ \cdot \end{array} \begin{array}{c} \times \\ \times \end{array} \cdot \quad \text{---} \quad \begin{array}{c} : \\ : \end{array} \text{---} \quad \begin{array}{c} : \\ : \end{array}$$

Some teachers may prefer the following arrangement (which has the same kind of symmetry as regards the figures), though it is not so convenient as regards spacing:

$$(d) \quad \begin{array}{r} 78.364 \times 359.6 \\ = 7826.4 \times 3.596 \end{array}$$

$$(e) \quad \begin{array}{r} 853.1 \times 0.00543 \\ = 0.8531 \times 5.43 \end{array}$$

For the arrangement of the work I personally prefer the following:

$\begin{array}{r} 7826.4 \\ 3.596 \\ \hline 23479.2 \\ 3913.20 \\ 704.376 \\ 46.9584 \\ \hline 28143.7344 \end{array}$	$6.923 \left\{ \begin{array}{l} 123.45 \\ 69.23 \\ 54.220 \\ 48.461 \\ 5.7590 \\ 5.5384 \\ 22060 \\ 20769 \\ \hline 1291 \end{array} \right.$	<p>or of course:</p> $\begin{array}{r} 17.83 \\ 6.923 \left\{ \begin{array}{l} 123.45 \\ 54.220 \\ 5.7590 \\ 22060 \\ 1291 \end{array} \right. \end{array}$
--	---	---

An alternative method for multiplication is to put *both* numbers in standard form. The work may then be shown thus:

$(f) \quad \begin{array}{rcl} 78.06 & = & 7.806 \times 10^1 \\ 0.0003427 & = & 3.427 \times 10^{-4} \\ \hline & & 23.418 \\ & & 3.1224 \\ & & \cdot 15612 \\ \hline & & 54642 \\ & & \overline{26.751162} \times 10^{-3} \\ & & = 0.026751162 \end{array}$	$\begin{array}{rcl} 7.806 \times 10^1 & & \\ 3.427 \times 10^{-4} & & \\ \hline 23.42 & & \\ 3.12 & & \\ \cdot 16 & & \\ \hline 5 & & \\ \hline 26.75 \times 10^{-3} & & \\ = 0.02675 & & \\ \hline \text{Ans. } 0.0268 & & \end{array}$
--	--

To a pupil who has no acquaintance with indices these may be explained simply as the number of times we must multiply, or (in the case of negative indices) divide, by ten.

To obtain an answer correct to (say) three significant figures, rule a vertical line after the third significant figure in the first partial product (this can be done before this first product is obtained), and work one place beyond this limiting line.

To obtain an answer correct to four decimal places, first add the indices of the two powers of ten (giving in this case -3). Add this to the number of decimal places required (giving $4 - 3 = 1$), and the result gives the number of decimal places in our working which must fall to the left of the "limiting line."

The above suggestions are made for those teachers who would prefer to retain standard form if possible, or who are using it now and consider it inadvisable to make an abrupt change in their higher or middle classes.

The method about to be described was worked out two years ago, while I was teaching a private pupil. Unfortunately I am not now engaged in school teaching, so have no opportunity of testing the method with classes of pupils. When I last taught a class I used standard form, as shown above (but without the additional suggestions illustrated in examples (a), (b), (c), (d), (e), (f), keeping all the decimal points (excepting of course that of the divisor) in one vertical column. The new method, however, appeals to me by its simplicity, and by the fact that it requires no change of method or arrangement from the beginning of multiplication and division up to the complete mastery of contracted methods. I am hoping, therefore, that those teachers who still find cause for dissatisfaction with the methods they have used in teaching will give this method a trial, if possible teaching it to their youngest classes, and noting the results at each stage in their progress.

The method is most easily followed by reference to the examples. These have been chosen solely with a view to illustrating the method in its various stages, and not as suitable first examples for pupils.

MULTIPLICATION OF WHOLE NUMBERS.

(1)	(2)	(3)	(4)	(5)
329×4	329×40	329×43	3724×541	2126×302
329	329	329	3724	2126
4	40	43	541	302
<u>1316</u>	<u>13160</u>	13160	18620	63780
		987	14896	4252
		<u>14147</u>	3724	<u>642052</u>
			<u>2014684</u>	

Beginning with multiplication: the arrangement in Example (1) is identical with the one commonly used.

Before learning long multiplication the pupil should realise clearly that 40 stands for "four times ten," and that this is the same as "ten times four," and also that the mere placing of a nought after a row of figures has the effect of multiplying the whole number by ten. It is then clear that multiplying by 40 is exactly the same as multiplying by 4 and placing a nought after the result. Thus the sum is written exactly as if we were multiplying by 4, and the nought placed to the right of the 4 demands a nought underneath it to the right of the product.

Ex. 3. Since 43 means $40 + 3$, the arrangement is precisely the same for multiplication by 43 as for multiplication by 40, but we must now add another row of figures representing 3 (not 30) times the multiplicand.

Ex. 4. Our rule for multiplication of whole numbers is now clear. Write down the multiplicand; write the multiplier underneath it, with the first figure under the *units* figure of the multiplicand. Multiply by the *first* (the most important) figure of the multiplier first, placing each figure of the partial product in the same vertical column as the figure we are multiplying. Multiply by each figure in succession, from left to right, each time beginning our partial product one place further to the right, so that in each row the figure resulting from multiplication of the units figure of the multiplicand comes in the same column as the figure in the multiplier which produces that row.

It is of course pointed out to the pupil that the "importance" of any figure depends on which column it appears in, so that it is not necessary to write the noughts after each row to indicate the value of that row, but merely to place the figures of each row in correct relative positions. Accordingly the nought, which in Example 3 has been placed at the end of the first partial product, is only used in the first instance, in order to make quite clear to the pupil the reason for placing each partial product one place further to the right, and in all subsequent work these noughts are omitted. When, however, a nought occurs in the multiplier, as in Example 5, a nought is written after the row we have just finished, and the next row begins of course one place to the right of this nought.

It should be noted that this method of placing the two factors to be multiplied shows at a glance how many figures there will be in the product—the product "stretches over" both the factors, coming one column further to the left if the product of the first figures of the two factors is a number of two figures.

MULTIPLICATION OF DECIMALS.

(6)	(7)	(8)
329.71×43.08	786.2×0.000127	0.00683×572.1
329.71	786.2	0.00 683
43.08	0.0001 27	5 72.1
<hr/>	<hr/>	<hr/>
1318 8.4	-0786 2	3.415
98 9.130	157 24	4781
2 6.3768	55 034	1366
<hr/>	<hr/>	<hr/>
1420 3.9068	0.0998 474	683
		<hr/>
		3.907443

If the factors to be multiplied contain decimal fractions as well as whole numbers (Example 6), the arrangement is exactly the same as before so far as the whole numbers are concerned, and the decimal figures simply follow on, care being taken to place all figures correctly in columns. Multiplication by the first figure of the multiplier (which, as before, is under the *units* figure of the multiplicand, yields figures which are placed in the same column as the figures in the multiplicand which produce them, and each partial product comes one place further to the right than the preceding, any noughts which occur in the multiplier giving rise to additional noughts at the end of the row just finished, thus lengthening that row and ensuring that the next row is correctly placed. The decimal point in the product is always under the decimal point in the multiplier.

Examples 7 and 8. If one or both factors contain a decimal fraction without a whole number, the rule is still the same:

RULE. Place the multiplier with its first significant figure (that is, the first figure other than a nought) under the **UNITS** figure of the multiplicand

(even if the units figure happens to be a nought, as in Example 8). The first partial product is placed with its figures under the corresponding figures of the multiplicand and the decimal point in the product is under the decimal point in the multiplier.

CONTRACTED MULTIPLICATION.

If we require a product correct only to (say) two decimal places, we rule a vertical line after the second decimal place in our multiplier, and work our partial products to one place beyond this line. This is effected by first crossing off any figures in the multiplicand which fall to the right of the last column of working. Multiplication by the first (significant) figure of the multiplier then gives a row correctly placed with its last figure in our last working column, and successive rows are prevented from going too far to the right by placing a dot over successive figures in the multiplicand, taking care to place a dot whenever a nought occurs in the multiplier.

To limit the number of significant figures in the product, we place our vertical line after this number of figures in the multiplicand, or one less if the product of the two first figures is 10 or more.

In obtaining the figures in the last column, account should always be taken of the "carrying figure" in respect of the last figure crossed off or dotted (or alternatively the work may be taken two columns beyond the limiting line and carrying figures ignored).

$$\begin{array}{r}
 (9) \\
 281.62 \times 0.056435 \\
 \text{to 4 significant figures} \\
 \text{or 2 decimal places.} \\
 \begin{array}{r}
 \dot{2} \ 81 \ .\dot{6}2 \\
 0.05 \ 6435 \\
 \hline
 14.08 \ 1 \\
 1.69 \ 0 \\
 .11 \ 2 \\
 8 \\
 1 \\
 \hline
 15.89 \ 2 \\
 \hline
 \text{Ans. } 15.89
 \end{array}
 \end{array}$$

SHORT DIVISION.

There seems nothing to be gained by any alteration in the ordinary method of setting out short division and division by factors. The work will therefore appear as in Examples 10, 11 and 12.

$$\begin{array}{r}
 (10) \\
 13173 \div 40 \\
 40 \overline{) 13173} \\
 \underline{3291\frac{3}{4}}
 \end{array}$$

$$\begin{array}{r}
 (11) \\
 85373 \div 42 \\
 6 \overline{) 85373} \\
 7 \overline{) 14228\frac{1}{2}} \\
 \underline{2032\frac{1}{2}}
 \end{array}$$

$$\begin{array}{r}
 (12) \\
 85.373 \div 42 \\
 \text{correct to 3 dec. places.} \\
 6 \overline{) 85.373} \\
 7 \overline{) 14.229} \\
 \underline{2.033}
 \end{array}$$

The remainder, 29, in Example 11, is obtained by the following mental work :

7 into 18=2, leaving 4, $\times 6 + 5 = 29$, over $6 \times 7 = 42$.

This method may be continued for any number of factors.

For long division, however, we revert to the arrangement already described for multiplication.

LONG DIVISION.

It should have been previously explained to the pupil, and should now be pointed out to him afresh, that multiplication is merely a short way of adding the same quantity a number of times, the "product" being the sum or result of such additions. In exactly the same way division is merely a short method of subtracting the same quantity a number of times from a given quantity (the dividend) and the answer may either be the remainder after subtracting the

"divisor" a certain number of times, or it may be the "quotient," that is, the "number of times" we are able to subtract the divisor from the dividend.

It would be inconvenient to place the divisor under the dividend as in the case of ordinary subtraction; we therefore place it *over* the dividend as far to the left as will not make it appear bigger than the dividend. A line is left blank between the divisor and the dividend, and in this blank space we place the figures of the quotient as they are obtained. The first figure of the quotient is placed *under the units figure* of the divisor, and the decimal point in the quotient comes immediately over the decimal point in the dividend. As in multiplication the first subtrahend is placed with its figures in the same columns as the corresponding figures of the divisor (which takes the place of the multiplicand), and each successive subtrahend is shifted one place further to the right.

$$\begin{array}{r}
 (13) \\
 14147 \div 329 \\
 \text{Divisor} \quad - \quad 329 \\
 \text{Quotient} \quad - \quad 43 \\
 \text{Dividend} \quad - \quad 14147 \\
 \quad \quad \quad 1316 \\
 \quad \quad \quad \hline
 \quad \quad \quad 987 \\
 \quad \quad \quad 987 \\
 \quad \quad \quad \hline
 \end{array}$$

$$\begin{array}{r}
 (14) \\
 2014684 \div 3724 \\
 \quad \quad 3724 \\
 \quad \quad \quad 541 \\
 \quad \quad \hline
 \quad 2014684 \\
 \quad \quad 18620 \\
 \quad \quad \hline
 \quad \quad 15268 \\
 \quad \quad \quad 14896 \\
 \quad \quad \quad \hline
 \quad \quad \quad 3724 \\
 \quad \quad \quad 3724 \\
 \quad \quad \quad \hline
 \end{array}$$

$$\begin{array}{r}
 (15) \\
 642052 \div 2126. \\
 \quad \quad 2126 \\
 \quad \quad \quad 302 \\
 \quad \quad \hline
 \quad 642052 \\
 \quad \quad 6378 \\
 \quad \quad \hline
 \quad \quad 4252 \\
 \quad \quad \quad 4252 \\
 \quad \quad \quad \hline
 \end{array}$$

$$\begin{array}{r}
 (16) \\
 14203.9068 \div 329.71 \\
 \quad 329.71 \\
 \quad 43.08 \\
 \quad \hline
 \quad 14203.9068 \\
 \quad 13188.4 \\
 \quad \hline
 \quad \quad 1015.50 \\
 \quad \quad 989.13 \\
 \quad \quad \hline
 \quad \quad 263768 \\
 \quad \quad 263768 \\
 \quad \quad \hline
 \end{array}$$

$$\begin{array}{r}
 (17) \\
 0.0998474 \div 786.2 \\
 \quad 786.2 \\
 \quad 0.000127 \\
 \quad \hline
 \quad 0.0998474 \\
 \quad \quad 786.2 \\
 \quad \quad \hline
 \quad \quad 212.27 \\
 \quad \quad 157.24 \\
 \quad \quad \hline
 \quad \quad 55.034 \\
 \quad \quad 55.034 \\
 \quad \quad \hline
 \end{array}$$

$$\begin{array}{r}
 (18) \\
 3.907443 \div 0.00683 \\
 \quad 0.00683 \\
 \quad 572.1 \\
 \quad \hline
 \quad 3.907443 \\
 \quad 3.415 \\
 \quad \hline
 \quad \quad 4924 \\
 \quad \quad 4781 \\
 \quad \quad \hline
 \quad \quad 1434 \\
 \quad \quad 1366 \\
 \quad \quad \hline
 \quad \quad 683 \\
 \quad \quad 683 \\
 \quad \quad \hline
 \end{array}$$

When the division is completed, it will be seen that the work and setting-out are identical with those in the corresponding multiplications, except that in multiplication we *add* the partial products *en bloc* in order to obtain the complete product, while in division we *subtract* them successively from the dividend. Examples 3 to 8 should be compared respectively with 13 to 18, when it will be seen that the first, second and last rows of figures in the multiplications are identical with the first, second and third rows in the divisions (except, of course, when there is a final remainder, in which case the dividend differs slightly from the product), and that the successive partial products are identical with the successive subtrahends. Division is thus clearly shown to be what it really is: the inverse of multiplication.

CONTRACTED DIVISION.

To obtain the quotient correct to a given number of *significant figures* without unnecessary work, rule a vertical line after this number of figures in the *divisor*, and work one place further to the right, crossing off or dotting successively any figures in the divisor which, when multiplied, would carry the work too far to

the right. For every figure dotted there must of course be a figure in the quotient, a nought if necessary.

To obtain a result correct to (say) four *decimal places*, the decimal point and the first significant figure must first be inserted in the quotient, in order to ascertain how many significant figures are needed to provide the required number of decimal places. The vertical line is then drawn after this number of significant figures in the divisor, as before.

It should be noted that the relative position of the vertical line and the figures of the second row (the quotient) has no significance; sometimes the quotient will go beyond the rest of the working, sometimes it will fall short of it. Exactly the same thing, of course, happens in the case of multiplication; it is the first row and the third and subsequent rows which are affected by the limiting line, the second row is quite independent of it.

In the examples of division shown above, the subtrahends have all been inserted, for the sake of comparison with the corresponding multiplications. I think that in the beginning the pupil should work a number of examples in this way, comparing the multiplications and the corresponding divisions, until he thoroughly realises the reciprocal nature of multiplication and division. He may then be shown the "Italian" method of division, whereby the subtrahends are omitted from the written work, and the multiplication and subtraction performed in one operation. The appearance of the work is then as in Example 20.

$$\begin{array}{r}
 (19) \\
 15.892 \div 281.62 \\
 \text{to 3 significant figures} \\
 \text{or 4 decimal places.} \\
 \hline
 281 \cdot 62 \\
 0.05 \ 643 \\
 \hline
 15.89 \ 2 \\
 14.08 \ 1 \\
 \hline
 181 \ 1 \\
 169 \ 0 \\
 \hline
 12 \ 1 \\
 11 \ 2 \\
 \hline
 9 \\
 8 \\
 \hline
 1
 \end{array}$$

Ans. 0.0564

Multiplication	(20)	Division
$ \begin{array}{r} 581 \cdot 73 \\ 0.0002 \ 7168 \\ \hline \cdot 1169 \ 5 \\ 409 \ 3 \\ 5 \ 8 \\ 3 \ 5 \\ 4 \\ \hline 0.1588 \ 5 \end{array} $		$ \begin{array}{r} 581 \cdot 73 \\ 0.0002 \ 716 \\ \hline \cdot 1588 \ 5 \\ 419 \ 0 \\ 9 \ 7 \\ 3 \ 9 \\ 4 \\ \hline \end{array} $
<u>Ans. 0.1589</u>		<u>Ans. 0.000272</u>

GENERAL REMARKS.

Care must be taken throughout the work to keep the figures correctly placed in columns. In order to avoid any inconvenience which might result from the non-correspondence of the decimal points in the first two rows, it is well to space the figures somewhat widely in the horizontal direction, so that the columns separated by decimal points are sensibly the same distance apart as any other pair of consecutive columns.

It will be noticed that the method described above proceeds by natural steps from the simplest multiplication, which is arranged exactly as in the older method, to multiplication and division of decimals, with or without contraction, and that the rule laid down in the first instance is not changed in the smallest particular, but is merely extended as required.

I am informed by Miss F. A. Yeldham, that, as regards the division of whole numbers, the method is practically identical with that described by Al Kho-

warizmi, an Arabian writer of the early ninth century, whose work on Arithmetic was the first to be translated into the English language. The only difference apparently was that Al Khowarizmi, after the Arabian fashion, placed his divisor *under* the dividend and worked upwards.

The correspondence of the three principal rows in multiplication and division suggests the desirability of a common name for each row. Each row has three names according to whether we are thinking of multiplication, division, or the value of a fraction. Thus :

The First Row is either	Multiplicand,	Divisor, or	Denominator.
Second Row	Multiplier,	Quotient,	Value of Fraction.
Third or Last Row	Product,	Dividend,	Numerator.

Since the first row is in each case as it were the *unit* with which we are dealing (in the case of the fraction it is the reciprocal of the unit), I suggest the name "denominator."

The second row is the "number of times we are adding or subtracting the first row; hence the name "Quotient" seems suitable. Owing to its alternative positions it is perhaps well to keep for the third or last row its double name of "Product" or "Dividend" (more especially as these two are not identical when there is a remainder), though by analogy with the first row it might be called the "Numerator."

If the above names be considered objectionable, we may perhaps refer simply to the "first and second rows."

R. A. M. KEARNEY.

756. On the whole, I must (according to my present lights) claim for Satan a freedom from all scientific restraints. This freedom is exemplified by his showing all the kingdoms of the world from an exceeding high mountain, thus affording the first practical demonstration of the flat-earth theory. . . —R. A. Proctor, *Myths and Marvels of Astronomy*, 1880, f.n. p. 257.

757. On ne peut lui refuser la justice de remarquer que personne avant lui ne s'est porté dans cette recherche [quadrature of the circle] avec autant de génie, et même, si nous en exceptons son objet principal, avec autant de succès.—Montucla, of Gregory St. Vincent, *Histoire des Recherches sur la Quadrature du Cercle*, 1754, p. 66.

758. Of Desargues and his general ability Descartes had to the end of his life a very high opinion. Asking for criticisms of the *Méditations Métaphysiques*, he wrote: "Je ne serais point fâché que M. des Argues fut aussi l'un de mes juges, s'il luy plaisoit d'en prendre la peine, et je me fie plus en lui seul qu'en trois théologiens." The acknowledgment of indebtedness of Pascal to Desargues, made by the youthful Blaise in his first contribution to the literature of conics, will be remembered by readers of the *Gazette*, vol. xii., March 1924, p. 54. It was no discredit to Pascal that Descartes attributed to Desargues the inspiration that produced *l'Essay pour les coniques*. Descartes met Desargues at the siege of La Rochelle, where the latter was engaged as an engineer. The two had originally been drawn together by a community of interest "au moyen de perfectionner la mécanique pour abrégier et adoucir les travaux des hommes." As Baillet puts it: "pour ne rendre pas inutile au public la connoissance qu'il avoit des mathématiques et particulièrement de la mécanique, il employoit particulièrement ses soins à soulager les travaux des artisans par la subtilité de ses inventions."—*Œuvres de Desargues*, edn. Poudra, 1864, vol. 2, p. 118.

759. La découverte du principe absolu, dont Euler avait reconnu la nécessité et que Laplace avait tenté d'établir, de ce principe qui embrasse l'ensemble tout entier de la science, et dont on peut déduire toutes les vérités mathématiques connues ou inconnues, a été définitivement réalisée per Hönen Wronski. —Monferrier, *Encyc. Mathématique*, 1856-9, p. xi [Sotheran, Cat. 816, No. 699].

MATHEMATICAL NOTES.

953. [K¹. 2. d.] *A generalisation of Feuerbach's Theorem.*

Feuerbach's Theorem, that the in- and escribed circles of a triangle touch the nine-points circle is a special case of the following:

If a conic touch the sides of a triangle, its joint-director circle with either confocal drawn through the circumcentre touches the nine-points circle.

I use the term *joint-director circle* of a conic $\frac{x^2}{A} + \frac{y^2}{B} = 1$ with a confocal

$$\frac{x^2}{A+\theta} + \frac{y^2}{B+\theta} = 1$$

to denote $x^2 + y^2 = A + B + \theta$, which is known to be the locus of intersection (when geometrically possible) of orthogonal tangents drawn one to either conic.

Let the circumradius be taken as unity, the circumcentre O as origin, and the complexes of the angular points as α, β, γ . Let $p = \Sigma\alpha, q = \Sigma\beta\gamma, r = \alpha\beta\gamma$.

The equation of the side $\beta\gamma$ is $z + \bar{z}\beta\gamma = \beta + \gamma$, and the perpendicular on it from z is $(z - \beta - \gamma + \bar{z}\beta\gamma)/2\sqrt{\beta\gamma}$.

If z_1, z_2 are complexes of the foci F_1, F_2 , and A, B are the squares of the semi-axis of the conic, then

$$\begin{aligned} 4B\beta\gamma &= (z_1 - \beta - \gamma + \bar{z}_1\beta\gamma)(z_2 - \beta - \gamma + \bar{z}_2\beta\gamma) \\ &= z_1z_2 - (z_1 + z_2)(p - \alpha) + (z_1\bar{z}_2 + z_2\bar{z}_1)\beta\gamma + (p^2 - p\alpha - q + \beta\gamma) \\ &\quad - (\bar{z}_1 + \bar{z}_2)(p\beta\gamma - r) + \bar{z}_1\bar{z}_2(q\beta\gamma - r\bar{p} + r\alpha), \end{aligned}$$

with two other similar equations got by cyclic change in α, β, γ .

This implies that we may equate to zero the coefficients of $\alpha, \beta\gamma$, and the symmetrical terms.

$$\begin{aligned} \text{Hence } 4B &= 1 + z_1\bar{z}_2 + \bar{z}_1z_2 - p(\bar{z}_1 + \bar{z}_2) + q\bar{z}_1\bar{z}_2, \\ z_1 + z_2 - p + r\bar{z}_1\bar{z}_2 &= 0, \\ z_1z_2 - q + r(\bar{z}_1 + \bar{z}_2) &= 0. \end{aligned}$$

Let M, N be respectively the centres of the conic and the nine-points circle. The complex of M is $\frac{1}{2}(z_1 + z_2)$, and since the centroid is $\frac{1}{3}p$, then N is $\frac{1}{2}p$.

$$\text{Thus } MN = \frac{1}{2} |z_1 + z_2 - p| = \frac{1}{2} |r\bar{z}_1\bar{z}_2| = \frac{1}{2}\rho_1\rho_2$$

where

$$\begin{aligned} \rho_1, \rho_2 &= OF_1, OF_2, \\ \text{Also } 4B &= 1 + z_1\bar{z}_2 + \bar{z}_1z_2 - (\bar{z}_1 + \bar{z}_2)(z_1 + z_2 + r\bar{z}_1\bar{z}_2) + \bar{z}_1\bar{z}_2(z_1z_2 + r(\bar{z}_1 + \bar{z}_2)) \\ &= 1 - z_1\bar{z}_1 - z_2\bar{z}_2 + z_1\bar{z}_1z_2\bar{z}_2 = 1 - \rho_1^2 - \rho_2^2 + \rho_1^2\rho_2^2. \end{aligned}$$

If $4(A + \theta)$ is the squared major-axis of the confocal ellipse through O , then $4(A + \theta) = (\rho_1 + \rho_2)^2$, so that $4(A + B + \theta) = (1 + \rho_1\rho_2)^2$.

Hence the radius of the joint-director circle with the given conic is $\frac{1}{2} + MN$. But $\frac{1}{2}$ is the radius of the nine-points circle, so that the circles touch.

Similarly for the confocal hyperbola $4(A + \theta) = (\rho_1 - \rho_2)^2$ and

$$4(A + B + \theta) = (1 \sim \rho_1\rho_2)^2,$$

and the same deduction applies as before.

Special Case. If O lies on the axis joining the real foci, then one confocal reduces to the two foci, and the joint-director circle is the auxiliary circle. This includes the cases when the conic is a circle, inscribed or escribed, for every diameter is a major axis. As these circles are their own auxiliary circles, they all touch the nine-points circle (Feuerbach).

The above theorem is, I believe, new. I have been unable to trace it in any books of recorded results, and it is unknown to various geometers of experience whom I have consulted.

I make no apology for my method. It is the one by which I hit upon the result, and no doubt pure geometric methods will be forthcoming and will be preferred by some geometers. I have all respect for pure geometry, but have

no head for it. It seems to me to require firstly a knowledge of the result to be obtained, and secondly an acquaintance with various theorems which are (to me at least) recondite. I have based the whole proof on the elementary fact that the product of perpendiculars drawn from the foci on a tangent to a central conic section is constant, ultimately applicable, of course, to parabolic cases. If any step in my analysis is recognisable as a known theorem by authors living or dead, I offer apologies to them for omitting their names. The use of complexes facilitates and abbreviates the work very considerably. A possible method of referring the central conic to its principal axes, and taking these tangents in terms of eccentric angles, requires reduced formulæ for their circumcentre and their nine-points centre, neither of which is quotable as a standard formula, and both are very tedious to work out. Methods by areals or tangential coordinates also present many difficulties.

L. J. ROGERS.

954. [v. 6. 7.] *Hallam, an authority on the History of Mathematics.*

Hallam was a very competent mathematician, in every matter which could concern an historian of the middle ages, up to the middle of the seventeenth century at least. A matterer is always erroneous when he attempts to be brief; but Hallam is as correct in his pithy notices of algebra as in any part of his work. . . .

[In his *Literature of Europe during the Middle Ages* he declines to enter on the period 1650-1700.] His chief reason, in his own words, is "the slightness of my own acquaintance with subjects so momentous and difficult, and upon which I could not write without presumptuousness and much peril of betraying ignorance. The names, therefore, of Wallis and Huyghens, Newton and Leibnitz, must be passed with distant reverence." *Expressio unius est exclusio alterius*: Hallam implies that, up to 1650, he writes without sense of presumption or fear of self-exposure: and he had no reason for either. Were I to write the history of Mathematics, I should certainly look on Hallam as one of the writers of authoritative opinion whom I should be glad to cite in my favour, and bound to oppose with reason when I differed from him.—A. De Morgan [*Athenæum*, 1859, p. 188, and v. p. 254].

955. [J. 2. f.] *A Case of Local Probability.*

A golf ball, driven from the tee, comes to rest on a perfectly square and level green, with no banks; the hole being exactly in the centre of the green. Nothing being assumed as to the player's skill, what is the chance that he will find his ball nearer to the pin than to any edge of the green?

Let S be the centre of a square $EFGH$; let X_1, X_2 be the mid-points of the sides EF, HE ; and let P be any point in the quarter square EX_1SX_2 , which area suffices for the purposes of the problem. We have to find the chance that P is nearer to S than to either EX_1 or X_2E .

Bisect SX_1, SX_2 in A_1, A_2 ; and with vertices A_1, A_2 , and common focus S , suppose two parabolas described towards E , intersecting on SE (as they must) in Q , say. Then if P lies within the space A_1QA_2S it is nearer to S than to either of the directrices EX_1, EX_2 ; but if it is outside the space A_1QA_2S it is nearer to one or other directrix, or possibly to both, than to S . The chance of equidistance, when P is on either arc A_1Q, A_2Q , may be disregarded.

From Q let fall perpendiculars QN_1, QN_2 on SX_1, SX_2 ; and denote A_1S by 1; A_1N_1 by x ; QN_1 by y .

Then $QN_1^2 = 4A_1N_1 \cdot A_1S = 4x$.

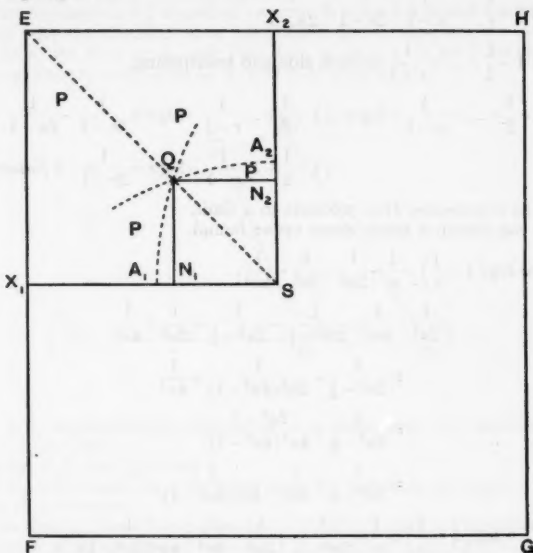
Also $QN_1^2 = \frac{1}{2}SQ^2$; but $SQ = N_1X_1$; $\therefore QN_1^2 = \frac{1}{2}X_1N_1^2 = \frac{1}{2}(1+x)^2$;

$\therefore x^2 - 6x + 1 = 0$; from which $x = 3 - 2\sqrt{2}$.

Area $A_1QN_1 = \frac{2}{3}xy = \frac{2}{3}x\sqrt{x} = \frac{2}{3}(3 - 2\sqrt{2})^{\frac{3}{2}} = \frac{2}{3}(\sqrt{2} - 1)^3 = \frac{2}{3}(5\sqrt{2} - 7)$;

„ A_2QN_2 is the same;

Area $QN_1SN_2 = QN_1^2 = 4x = 4(3 - 2\sqrt{2})$;
 „ $EX_1SX_2 = 4$.



Hence the required chance

$= [(A_1QN_1) + (A_2QN_2) + (QN_1^2)] / EX_1SX_2 = \frac{2}{3}(5\sqrt{2} - 7) + (3 - 2\sqrt{2}) = 0.2189$
 i.e. the chance is approximately 127/576, being greater than 7/32 but less than 2/9.

G. WOTHERSPOON.

956. [D. 2. b.] *A close upper bound for Euler's Constant.*

In what follows letters denote integers, and the base of logarithms is e .

Let $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n-1} - \log n \equiv f(n)$, and $\lim_{n \rightarrow \infty} f(n) = C$.

We have $\log\left(1 + \frac{1}{n}\right) < \frac{1}{n}$.

Hence $\log \frac{n}{n-1} < \frac{1}{n-1}$, $\log \frac{n-1}{n-2} < \frac{1}{n-2}$, ..., $\log \frac{r+1}{r} < \frac{1}{r}$.

Hence, by addition, $\log n - \log r < \frac{1}{r} + \dots + \frac{1}{n-1}$.

Adding $1 + \frac{1}{2} + \dots + \frac{1}{r-1}$ to each side and transposing, we have $f(n) > f(r)$, so that $f(n)$ is an increasing function.

Again, $\log\left(1 + \frac{1}{n}\right) > \frac{1}{n} - \frac{1}{2n^2}$
 $> \frac{1}{n} - \frac{1}{2n^2 - 1}$, *à fortiori*,

or $> \frac{1}{n} - \frac{1}{2n-1} + \frac{1}{2n+1}$;

Hence writing $n-1, n-2, \dots, r$ for n and adding we obtain

$$\log n - \log r > \frac{1}{r} + \dots - \frac{1}{n-1} - \frac{1}{2r-1} + \frac{1}{2n-1}.$$

Adding $1 + \frac{1}{2} + \dots + \frac{1}{r-1}$ to each side and transposing,

$$\begin{aligned} 1 + \frac{1}{2} + \dots + \frac{1}{n-1} - \log n &< 1 + \frac{1}{2} + \dots - \frac{1}{r-1} - \log r + \frac{1}{2r-1} - \frac{1}{2n-1} \\ &< 1 + \frac{1}{2} + \dots - \frac{1}{r-1} - \log r + \frac{1}{2r-1}, \text{ à fortiori.} \end{aligned}$$

Hence, as n increases, $f(n)$ proceeds to a limit.

But we can obtain a much closer upper bound.

$$\text{We have } \log\left(1 + \frac{1}{n}\right) > \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \frac{1}{4n^4}.$$

$$\begin{aligned} \text{Also } \frac{1}{2n^3} + \frac{1}{4n^4} &= \frac{1}{2n^2 - \frac{1}{2}} - \frac{1}{2n^2 - \frac{1}{2}} + \frac{1}{2n^3} + \frac{1}{4n^4} \\ &= \frac{1}{2n^2 - \frac{1}{2}} - \frac{1}{2n^2(4n^2 - 1)} + \frac{1}{4n^4} \\ &= \frac{1}{2n^2 - \frac{1}{2}} + \frac{2n^2 - 1}{4n^4(4n^2 - 1)} \\ &= \frac{1}{2n^2 - \frac{1}{2}} + \frac{1}{8n^4} - \frac{1}{8n^4(4n^2 - 1)}. \end{aligned}$$

$$\begin{aligned} \text{Hence, } \log\left(1 + \frac{1}{n}\right) &> \frac{1}{n} - \frac{1}{2n^2 - \frac{1}{2}} + \frac{1}{3n^3} - \frac{1}{8n^4} + \frac{1}{8n^4(4n^2 - 1)} \\ &> \frac{1}{n} - \frac{1}{2n^2 - \frac{1}{2}} + \frac{1}{3n^3} - \frac{1}{8n^4}, \text{ à fortiori,} \\ &> \frac{1}{n} - \frac{1}{2n^2 - \frac{1}{2}} + \frac{1}{3\left(n^3 + \frac{1}{4n}\right)} - \frac{1}{8\left(n^4 - \frac{1}{n^2}\right)}, \text{ à fortiori,} \end{aligned}$$

$$\begin{aligned} \text{or } &> \frac{1}{n} - \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) \\ &+ \frac{1}{3} \left(\frac{1}{2n^2-2n+1} - \frac{1}{2n^2+2n+1}\right) \\ &- \frac{1}{48} \left(\frac{2n-1}{n(n-1)(n^2-n+1)} - \frac{2n+1}{n(n+1)(n^2+n+1)}\right). \end{aligned}$$

Whence proceeding as before we have that

$$\begin{aligned} 1 + \frac{1}{2} + \dots + \frac{1}{n-1} - \log n &< 1 + \frac{1}{2} + \dots + \frac{1}{r-1} - \log r + \frac{1}{2r-1} \\ &- \frac{1}{3} \cdot \frac{1}{2r^2-2r+1} + \frac{1}{48} \cdot \frac{2r-1}{r(r-1)(r^2-r+1)}. \end{aligned}$$

E.g. when $r=2$, $C < 1 - \log 2 + \frac{1}{3} - \frac{1}{15} + \frac{1}{96}$ or $\cdot 58393$; and when $r=5$,

$$C < 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \log 5 + \frac{1}{9} - \frac{1}{123} + \frac{1}{2520} \text{ or } \cdot 57727.$$

The value of C is $\cdot 57721566\dots$

N. M. GIBBINS.

957. [V. I. a. μ .] *The Teaching of Relative Velocity.*

The following method of teaching Relative Velocity has found favour with some of my students and I therefore venture to submit it to the *Gazette*.

Let two particles A and B have velocities u and v respectively. At time t_0 (in any units) let A be at A_0 and B at B_0 ; at time $t_0 + 1$ let A be at A_1 and B at B_1 ; at time $t_0 + t$ let A be at A_t and B at B_t (Fig. 1).

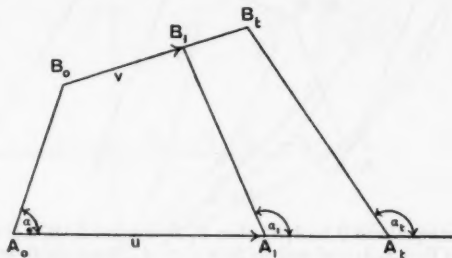


FIG. 1.

Then A_0A_1 represents the velocity of A in magnitude and direction and so $A_0A_1 = u$, $A_0A_t = ut$.

Similarly $B_0B_1 = v$, $B_0B_t = vt$.

Let A_0B_0 make an angle α_0 , A_1B_1 an angle α_1 , and A_tB_t an angle α_t with the direction of A 's motion.

Then at time t_0 A sees B at a distance A_0B_0 from it and in a direction α_0 with the direction of its motion, at time $t_0 + 1$ A sees B at a distance A_1B_1 from it and in a direction α_1 with its own line of motion, and at time $t_0 + t$ A sees B at a distance A_tB_t from it and in a direction α_t with its line of motion. These results are plotted in Fig. 2, where $AB'_0 = A_0B_0$, $AB'_1 = A_1B_1$ and $AB'_t = A_tB_t$,

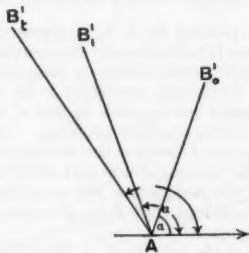


FIG. 2.

the angles which these lines make with A 's line of motion being α_0 , α_1 and α_t respectively. Thus the path of B as seen by A (or in other words the path of B relative to A) is some curve through the three points $B'_0B'_1B'_t$. The students are now asked to note the difference between the actual path of B ($B_0B_1B_t$ of Fig. 1) and the relative path in Fig. 2.

The questions which we must now determine are: (1) The nature of the path $B'_0B'_1B'_t$ and (2) the way in which this path is described by B as seen by A . These questions are best solved by superimposing Fig. 2 on Fig. 1, so

If $x^n = e^x$, then $x = e^{x/n}$.

Draw the straight line $y=x$ and the curve $y=e^x$.

By increasing the abscissa of any chosen point on $y=e^x$ in the ratio of $n:1$ and leaving the ordinate unaltered we can obtain a point on the curve $y=e^{x/n}$, e.g. from the point $P_1(1, e)$ on $y=e^x$ we have the point $P_2(2, e)$ on $y=e^{x/2}$, the point $P_3(3, e)$ on $y=e^{x/3}$, and so on. The curves $y=e^{x/2}$, $y=e^{x/3}$, etc., can thus be drawn with very little difficulty.

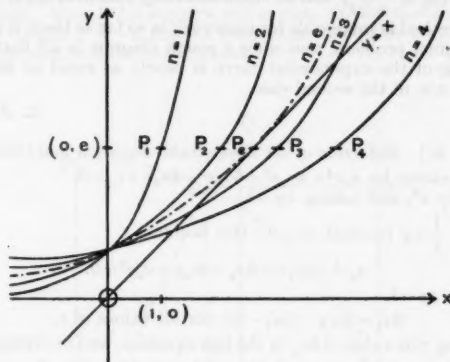


FIG. 1.

We notice that if $n \leq 2$ the curves $y=e^{x/n}$ do not meet the line $y=x$, whereas if $n \geq 3$ the curves $y=e^{x/n}$ all intersect the straight line in two distinct points, i.e. if $n \leq 2$, the equation $x^n = e^x$ has no real roots, and if $n \geq 3$, there are two real and distinct roots.

The further question then arises, for what value of n (between 2 and 3) does the equation $x^n = e^x$ have a repeated root, or, for what value of n (between 2 and 3) does the curve $y=e^{x/n}$ touch the line $y=x$?

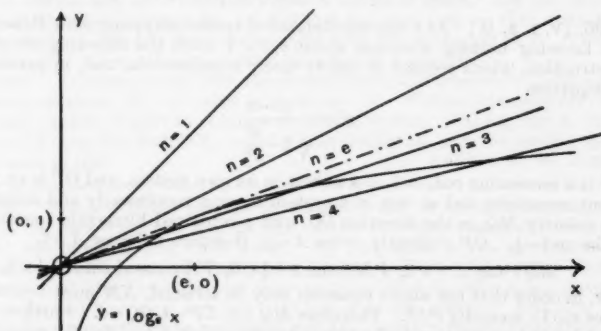


FIG. 2.

The obvious tentative answer is that $n=e$, the point of contact being the point (e, e) . This may be found directly, since $x^n = e^x$ will have a repeated root if $x^n = e^x$ and $nx^{n-1} = e^x$ can be simultaneously satisfied, i.e. if $x=n=e$.

The intersection of the curves $y=x^n$ and $y=e^x$ may also be obtained, and still more simply, from the intersection of $y=x/n$ and $y=\log_e x$. Again we find that if $n \leq 2$ there are no points of intersection, and if $n \geq 3$ there are two distinct points of intersection.

Also, if $n=e$, the line $y=x/n$ is a tangent to the curve $y=\log_e x$ at the point $(e, 1)$. This may be shown directly since $x/n=\log_e x$ has a repeated root if $x/n=\log_e x$, $\frac{1}{n}=\frac{1}{x}$, can be simultaneously satisfied, i.e. if $x=n=e$.

The first method is preferable for classwork in so far as there is less departure from the original problem, and since a rough diagram is all that is necessary the replotting of the exponential curve is nearly as rapid as the drawing of the lines $y=x/n$ in the second case.

E. J. TERNOUTH.

959. [A. 3. h.] *Reduction of the Biquadratic Equation to a Cubic.*

Let the equation be $a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4 = 0$.

Dividing by x^2 , and taking $xy=1$.

On letting $\frac{1}{x}=y$ (so that $xy=1$) this becomes

$$a_0x^2 + 4a_1x + 6a_2 + 4a_3y + a_4y^2 = 0.$$

Now, since $xy=1$,

$$6a_2 = 2zxy + (6a_2 - 2z) \text{ for all values of } z.$$

Substituting this value of $6a_2$ in the last equation, and re-arranging, we have

$$a_0x^2 + 2zxy + a_4y^2 + 4a_1x + 4a_3y + (6a_2 - 2z) = 0.$$

This breaks up into linear factors in x and y if

$$\begin{vmatrix} a_0 & z & 2a_1 \\ z & a_4 & 2a_3 \\ 2a_1 & 2a_3 & 6a_2 - 2z \end{vmatrix} = 0, \text{ the required cubic.}$$

By substituting $2a_0\theta + a_2$ for z , this cubic reduces to the cubic

$$4a_0^2\theta^3 - Ia_0\theta + J = 0.$$

C. W. HILDEBRAND.

960. [v. 1. a. β .] As a non-mathematical reader grappling with Relativity and knowing nothing whatever about $g_{\mu\nu}$'s, I made the following simplified construction, which seemed to satisfy many requirements, and, in particular, the equation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

OM is a measuring rod, $=1$, $=x$ at rest in its own system, and OP is an equivalent measuring rod at rest in a system moving rectilinearly and uniformly at a velocity MQ , in the direction OQ , with a contrived Fitzgerald contraction of the rod $=\frac{1}{2}$. OP evidently $=\cos A = \frac{1}{2}$, therefore $OQ = \sec A = 2$,

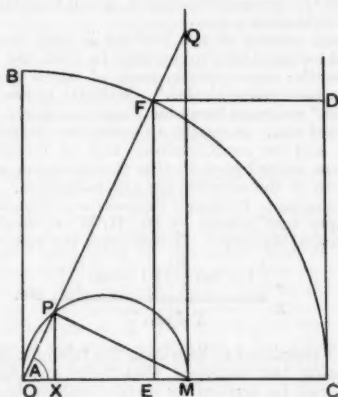
$$MQ = \tan A = \sqrt{3}, PM = \sin A = \frac{1}{2}\sqrt{3}, PX = \cos A \sin A = \frac{1}{4}\sqrt{3}.$$

Now, in order that the above equation may be satisfied, XM must be firstly vt (since $x=1$), secondly v^2/c^2 . Therefore $MQ=v$, $XP=t$, $OQ=c$. Further, MP , being twice t , $=$ (as result of sudden insight) clock (i.e. "double seconds"). (Vide Eddington, *Nature of the Physical World*, p. 54.)

The ratio $OQ : MQ$ is extremely important, for it is the ratio of the speed of light to the velocity of the moving system, and since, for any angle A , $\tan A$ can never be as long as $\sec A$, except at ∞ , it is seen at once in what manner c is a limiting velocity.

When the Fitzgerald contraction is $\frac{1}{2}$, the velocity of the moving system, with respect to the first, is, *ipso facto*, circa 161,000 m.p.s. and the speed of light (*vide op. cit.*), is 25,000 m.p.s. faster than this. If I take the speed of light to be exactly $(1 + \sqrt{3}/2) \cdot 10^8$ m.p.s. (186602.5), then the difference is *exactly* 25,000 m. Can this figure be taken, then, as the theoretical speed of light in, say "light-miles"?

Now, if we want to see ourselves as we look from the moving system, we invert the figure; then PO is the new rod at rest, XO the new Fitzgerald contraction, $OM=c$ and $PM=v$, t is the perp. from X on PO , and PX is clock. Hence the equivalence of the two systems.



To find graphically the Fitzgerald contraction for any velocity (?)

Let $OM=1$ and $OC=1.866025$ hundred thousand miles. Let OB, MQ, CD be perpendiculars. Draw the arc CB and the semicircle on OM . Mark the velocity along CD , and draw DF parallel with CO to the arc at F . Through F draw OQ . Then OP is the contraction for that velocity.

Prof. Empirical proof from the circumstance that when $OP = \frac{1}{2}$ the correct velocity of about 161,000 miles is given on CD . Further, the ratio $OF/FE = OQ/QM$ = speed of light to velocity of moving system. OQ appears to = mass. When OQ coincides with OM , velocity is evidently nil, and the moving system has come to rest.

ARNOLD J. W. KEFFEL.

ARNOLD J. W. KEPPEL.

760. "Satan delights equally in statistics and in quoting scripture. . . . 'Job,' he said, 'lived to a great age. After his disagreeable experiences he lived one hundred and forty years. He had again seven sons and three daughters, and he saw his offspring for four generations. So much is classical. These ten children brought him seventy grandchildren, who again prospered generally and had large families. (It was a prolific strain.) And now if we allow three generations to a century, and the reality is rather more than that, and if we take the survival rate as roughly three to a family, and if we agree with your excellent Bishop Ussher that Job lived thirty-five centuries ago, that gives us—How many? Three to the hundred and fifth power? . . . It is at any rate a sum vastly in excess of the present population of the earth. . . . You have globes and rolls and swords and stars here [i.e. in heaven]; has anyone a slide-rule.'"—From *The Undying Fire*, by H. G. Wells, ch. 1, § 3.

REVIEWS.

Le Probleme de Malfatti, le Pendule de Foucault et Autres Questions d'Analyse et de Physique. Par LOUIS GÉRARD. (Paris, Librairie Vuibert.)

In this little pamphlet Dr. Gérard has put together a number of short notes on topics ranging over a wide field. Several of the papers are of a character to interest, and be followed by, boys of Scholarship standard, and they could be used as a preparation for the reading of more elaborate papers in a foreign tongue. Two, perhaps, will appeal especially, the first, a method of approximating rapidly to e and π by means of certain continued fractions, the other a straightening up of the geometrical proof, given faultily in Deschanel, for minimum deviation through a prism.

The most important section of the booklet is that devoted to Malfatti's classical problem. As originally propounded in 1803 the problem is: "In a given triangle to inscribe three circles, each of which touches two sides of the triangle and the two other circles." Malfatti himself gave a solution, based on what he calls "scabrous formulae," and one might have thought that there the matter would end. Actually an extensive literature has grown up around the problem and its modifications, and as recently as May, 1928, Professor L. J. Rogers contributed to this *Gazette* what is undoubtedly the neatest analytical form of the solution for the basic case. (May the reviewer be here allowed to apologise to Professor Rogers for overlooking this important contribution in a paper sent jointly by Mr. H. W. Richmond and himself to the London Mathematical Society.) In this form the radii are given by

$$\frac{r}{2} \frac{\left(1 + \tan \frac{B}{4}\right) \left(1 + \tan \frac{C}{4}\right)}{1 + \tan \frac{A}{4}}, \text{ etc.}$$

If the circles are not restricted to lie within the triangle, there are thirty-two sets and the interesting fact emerges, that from any one solution all the remaining thirty-one can be derived by merely substituting for A, B, C , the angles of the triangle, other angles $A + \alpha\pi, B + \beta\pi, C + \gamma\pi$, where $\alpha + \beta + \gamma \equiv 0 \pmod{4}$ or $-A + \alpha'\pi, -B + \beta'\pi, -C + \gamma'\pi$, where $\alpha' + \beta' + \gamma' \equiv 2 \pmod{4}$.

The simplest case of the above transformations, viz. where A, B, C are changed into $-A, \pi - B, \pi - C$, was remarked by Lemoine in another connection, and could be made much use of in school trigonometry, for by means of it the in-circle is transformed into the ex-circle opposite to the vertex A , and formulae involving the ex-circles can be immediately derived from those of the in-circle. It is surprising that this is passed over in the text-books.

To revert to Malfatti, two extensions suggest themselves; the triangle may be replaced by circles in a plane, or by great circles on a sphere. Dr. Gérard attacks the first by means of the Darboux-Froebenius Identity for five circles, that if c_{12} is the cosine of the angle between the circles C_1, C_2 , then the symmetrical five-rowed determinant $(c_{11}, c_{22}, c_{33}, c_{44}, c_{55})$ is zero. He obtains an elegant solution much akin to that referred to above for the triangle. (In recent correspondence Prof. Rogers has treated the problem from the same point of view.) Schellbach's solution for the rectilinear triangle is included in this section as well as one for the spherical triangle.

Practically all the work on Malfatti's Problem and its ramifications is analytical, Steiner's beautiful geometrical solutions having presumably discouraged other attempts in this field. An interesting account of Steiner's constructions will be found in Coolidge's *Treatise on the Circle and Sphere*.

H. LOB.

The Adjustment of Errors in Practical Science. By R. W. M. GIBBS. Pp. 110. 5s. net. 1929. (Oxford Univ. Press.)

Extensive application of statistical methods to many fields of inquiry—to natural and social science, to psychology and pedagogy—has given rise to a

number of books in which the theory of such methods is given in more or less detail. But the "literature" of statistics gains in the present small volume an addition which we are glad to see. The right of the volume to welcome from the practical investigators for whom it is primarily intended rests upon the success with which the author has condensed into brief compass a clear account of the essential ideas that must be grasped by all who would use statistical methods intelligently and safely. The book will not be found easy on all its pages by "non-specialist" readers, to whom the author hopes it may make some appeal, and indeed such readers will find many larger books simpler. But the more advanced students whom the author has principally in mind are provided with chapters on frequency distributions, theory of errors, and correlation, which give the essentials with an admirable clearness which we unhesitatingly commend, and in a manner of which we do not think any mathematical reader will disapprove. Proofs which involve rather more difficult mathematics are put into a carefully written appendix. Only occasionally does the author write loosely, and then with no serious consequences, as when he speaks of "the centre of gravity of a column of dots" (p. 104), meaning "centre of mean position." We know of no other book which, for its size, contains so much that is at once useful to the experimental, and interesting to the purely theoretical, student of statistics. E. R. HAMILTON.

Elementary Applications of Statistical Method. By H. BANISTER. Pp. 56. 3s. 6d. net. 1929. (Blackie.)

This is a concise little book of about 50 pages, written by a psychologist as a kind of intelligent and persuasive handbook of the simpler forms of statistical technique—intelligent in the sense that it suggests reasons for doing things and persuasive in its manner of gently encouraging good habits. It deals with frequency distributions, means and measures of dispersion, the correlation coefficient—and includes the modern tests of significance for means of small samples and the use of χ^2 . But it is a pity that the position of these in a general scheme is not made more clear; for, although the existence of different types of frequency distribution is referred to at first, even the normal type is not clearly defined and the references to normality in connection with the tests are not very prominent. The treatment of correlation is unsatisfactory because of the deliberate omission of regression, which is much more useful in general than the coefficient r alone.

The book is neat and attractive, contains exercises and answers, some very pleasing graphical tables for testing, and an index which is rather inadequate even for so small a book, when its purpose is considered. G. SMEAL.

Mathematics Preparatory to Statistics and Finance. By G. N. BAUER. Pp. vi + 337. 8s. 6d. net. 1929. (Macmillan.)

This attempt at designing a course of elementary mathematics suitable for a special class of students is based on the author's experience at the University of New Hampshire, and is hardly likely to be entirely satisfactory for use in this country. But it is doubtful if any book better designed for the purpose yet exists, and, in recommending it to the notice of anybody concerned with preparation of non-mathematical students for courses in statistics and allied subjects, it seems worth while to comment briefly on its merits and deficiencies from this point of view. Indeed the chief merit of the book lies in the author's success in holding to the declared aim—what is included is there, not because it interested mathematicians, but because it contributes to the examination of phenomena. In the traditional sense the book may be said to deal with algebra and coordinate geometry, but coordinate geometry is treated as a method of studying relations between variates, whether those relations are originally algebraic or graphical in form. Thus in the chapter headings we have "the straight line law," "the law of the parabola," "the exponential law," and then, on the basis of these, "curve-fitting." Emphasis is rightly laid on methods of identifying curves and transforming them to straight lines, on the use of aids such as logarithmic paper, on the selection of parameters by least squares or otherwise. The algebraic portions are partly subservient

to this, partly concerned with interest and progressions, combinations and probability.

Even students of good mathematical antecedents would benefit by a preliminary course of this sort; but statistics, perhaps more than any other application of mathematics, is likely to be chosen by those who have no bent toward the mathematical or physical sciences. For these it is necessary to include an introduction to ideas which must not be left hazy if the course in statistics is to be anything more than arithmetic. In particular the idea of frequency for a *continuous* attribute requires a definite appreciation of the relation between a curve and its area, and, although some approach to this can be obtained without definite study of the calculus, it is hardly necessary now to urge that inclusion of the elements of the calculus is best in the long run. This would also improve considerably the treatment of the shapes of curves, of least squares and so on. The content of the course need not be increased to any great extent if such matters as frequency itself, and particularly correlation, are left to the statistical course, where they can be based on an adequate study of data.

In itself the book is an example of good production, with clear print, well-designed diagrams and tabular matter in the text, and many exercises. At the end are a table of 4-place logarithms, an index, and answers. G. SMEAL.

Elementary Trigonometry. By C. V. DURELL and R. M. WRIGHT. Pp. xviii + 288. Tables. Pp. 31. Answers. Pp. xxiv. 5s., or in three parts, price 2s. each. Parts I and II bound together, 3s. 6d. 1929. (G. Bell and Sons Ltd.)

This is an extremely thorough book with the clarity of explanation and wealth of illustration and example one always gets from Mr. Durell and his collaborators. The three parts are concerned with I. The Right-Angled Triangle, II. The General Triangle and Mensuration, III. The General Angle and Compound Angles; thus covering the School Certificate and Matriculation Stage.

The book begins with four pages of formulae and a good historical note. It is worth while to quote as briefly as possible the contents of the chapters in order to indicate the authors' idea of the development of the subject.

Part I. Chapter I. The Tangent of an Angle. II. The Sine and Cosine. III. Coscant, Secant, and Cotangent. IV. The Right-Angled Triangle. V. Three Dimensional Problems. VI. Graphical Methods.

Part II. Chapter VII. Angles greater than a Right Angle. VIII. Use of Logarithm Tables. IX. Solution of Triangles. X. Mensuration of the Circle. XI. Circular Measure. XII. Triangles and Polygons.

Part III. Chapter XIII. Identities involving Single Angles. XIV. The General Angle. XV. Compound Angles. XVI. Sums and Products Formulae. XVII. Half Angle Formulae. XVIII. Equations and Elimination. XIX. Miscellaneous Identities.

There are also six sets of revision papers equally spaced throughout the book, tables, and a complete set of answers.

One special feature of the book is that methods of solving the general triangle by division into right-angled triangles have been omitted, the authors believing that it is wrong policy to teach a method which will shortly be superseded. Another special feature is the use of Prof. Heath's proof of the addition theorem based on the idea of coordinates, with its complete generality. But for the sake of the weaker brethren among teachers and pupils the more usual proofs for restricted values of the angles have been given as an alternative. A third feature is the abundant use made of diagrams to illustrate the examples. This obviously saves much verbal description and affords scope for a much greater variety of examples than is usual.

These special features are not revolutionary in character, but are real improvements on the usual routine of teaching practice. But the further volume, which is promised and which will deal with Higher Certificate and Scholarship work, is distinctly revolutionary and extremely interesting (for the reviewer has seen it in proof), and it will be a very important contribution indeed to the practice of mathematical teaching.

N. M. G.

Höhere Mathematik für Mathematiker, Physiker und Ingenieure. Teil I. By R. ROTHE. Dritte Auf. Pp. 188. RM. 6. 1930. (Teubner.)

This first volume of Dr. Rothe's book is devoted to the exposition of the groundwork of the Calculus, and simple applications. The reasoning is rigorous, and an amazing amount of interesting matter is compressed into less than two hundred pages. This compression is attained not entirely by close printing, but also by a condensation in style which occasionally verges on obscurity. A schoolboy of "Scholarship Standard" would find much in this book to help him in manipulative technique, but the half-page on Dedekind sections, for example, would be of little value to him, even though the concept were not novel. An Honours student attending a set of "First Year" lectures on the Real Variable, reading the subject in, say, the fascinating pages of De La Vallée Poussin, might do well to remind himself of the more mundane aspects of the subject by returning to Dr. Rothe's volumes. Thus De La Vallée Poussin will tell him in a couple of lines that for the function

$$f(x, y) = xy(x^2 - y^2)/(x^2 + y^2), \quad f(0, 0) = 0,$$

we have

$$f_{x,y}(0, 0) = -1, \quad f_{y,x}(0, 0) = 1,$$

but in Rothe he will find half a page given to this example, and a diagram. Every point of the theory is driven home by a number of worked and unworked examples. In this lies the real value of the book, especially since the examples are chosen from a wide field ranging from elementary plane differential geometry to "black body" radiation. T. A. A. B.

Practical Mathematics for Juniors. By G. W. MANFIELD. Pp. vi + 126. 3s. 1928. (Blackie and Son, Ltd.)

This little book is divided into two parts.

Part I deals with brackets, simple equations, graphical solution of problems and equations, indices, logarithms, compound interest.

Part II presents an elementary experimental course in geometry and mensuration; simple problems on the polygon, cone, cylinder and sphere are solved, and the use of the trigonometrical functions is explained.

The author has managed to squeeze into this small volume a large amount of material. But the value of some of the bookwork would have been increased if a little more care had been used in its expression. For instance, we find,

$$\begin{aligned} \text{"Amount of money spent} &= 2 \times 4/5 + 3 \times 2/3 \\ \text{(Here } 4/5 \text{ is used for } 4s. 5d.); \end{aligned}$$

$$\text{and} \quad \frac{x}{3} \text{ cost } £9 \text{ each, i.e. } \frac{x}{3} \times 9 = £3x."$$

The method which the author uses to show that the area of a sphere is $4\pi r^2$ consists in the wrapping of a piece of string first round the sphere, and, secondly, round its circumscribing cylinder. Apart from the difficulty of performing this operation, the method is unsound, for the simple reason that the concept of Area does not depend on that of Length. The price is not unreasonable, and the book might be usefully employed in the teaching of either Central school pupils, or the best pupils in elementary schools.

The large number of practical examples should be useful for this purpose.

V. NAYLOR.

A Course in General Mathematics. By C. H. CURRIER and E. E. WATSON. Pp. viii + 413. 15s. net. 1929. (The Macmillan Company, N.Y.)

This book contains Algebra, Trigonometry, Analytical Geometry and the Calculus, pleasantly interwoven. The table of contents is very brief and does not do justice to the ground covered, which is as follows: functions and graphs, trigonometrical functions, logarithms, radian measure, straight line formulae, theory of equations, use of determinants, differentiation and integration, relations among trigonometrical functions, polar coordinates, progressions and binomial theorem, the exponential function, conic sections, space of three dimensions, permutations and combinations, theory of measurements, complex numbers.

At the outset, the authors express a desire to awake in the student his interest in the science of Mathematics, and, to this end, they give a short Historical résumé with each section treated. These little excursions cannot fail to help the student to realise that there are Histories other than the History of Kings, and that, in Mathematics, he has a science with great traditions behind it. The idea, which the authors have used, of mixing the ingredients, is a good one: nothing causes the student's energies to flag so much as the sight of a book with hundreds of pages in it, all dealing with the same subject. There is another reason why the book will be more to the liking of English readers than a good many American books: it is free from both archaisms and newly coined words. American idiom, however, occasionally obtrudes itself. As examples of this we take the following:

"If they are connected by a smooth curve, it will approximate the required parabola."

"Multiply the ten's digit by one more than itself and annex 25."

On the whole, the trigonometry is well treated. But one or two suggestions may be offered. To define the ratios of obtuse angles, the authors say: "if (x, y) (r, A) refer to a point in the second quadrant, then

$$\sin A = y/r, \cos A = x/r,"$$

and, later on they are forced to use " $\cos(180^\circ - A) = ON/OQ = -\cos A$," so that it is not clear where the negative sign has come from. It would be better to let (x_1, y_1) , (r_1, A) refer to a point in the first quadrant and $(-x_1, y_1)$, (r_1, A') with $A' = 180^\circ - A$, refer to a point in the second quadrant.

Then $\cos(180^\circ - A) = \cos A' = \frac{-x_1}{r_1} = -\frac{x_1}{r_1} = -\cos A$.

This supposes that, for $90^\circ > A > 0$ we define

$$\cos A \text{ and } \sin A \text{ by } \cos A = \frac{x_1}{r_1}, \sin A = \frac{y_1}{r_1}.$$

We then extend this definition by a process of generalisation; we define $\cos A$ and $\sin A$ in all other cases to be such that the equalities $\cos A = \frac{x}{r}$, $\sin A = \frac{y}{r}$ shall hold for all x, y, r and A , and this, be it noted, where r may lie in the range $(-\infty, \infty)$ and A in the range $(-2\pi, 2\pi)$.

By so doing (and this is the most important use of the trigonometrical functions) we insure that $\cos \theta$ and $\sin \theta$ shall be the operators by which we establish correspondence between the cartesian coordinates and the polar coordinates of any point. This, then, leads quite naturally to the treatment of projection and to the expansion of $\sin(A+B)$ by projection. The authors do not treat $\sin(A+B)$ by projection.

There is one other important point to which it is desired to direct attention. Throughout the book there is a bias towards Magnitude ("physical quantity") but the authors do not make it clear at what point the transition from Magnitude to numerical quantity is made.

We find such an equality as $s = r \cdot \theta$ (radians) $= 4.85 \times 10^{-6} \cdot \theta^\circ$. This should be written as either $s = r \cdot \theta$, or $s = 4.85 \times 10^{-6} \cdot r \cdot m$. The student is confused when (radians) is inserted to remind him that the θ is a measure in terms of the radian. If it is desired to mention the radian, the formula should be written $S = R \cdot \frac{\theta}{K}$, where K is the radian and capital letters refer to Magnitudes.

There is confusion of a similar kind on p. 332. Work is defined as "Force times component distance;" afterwards, on p. 333, we have

"if $F = 25$ lbs., and $x = 8$ feet, the work

$$W = 25 \times 8 = 200 \text{ foot-pounds}."$$

How, the student may ask, can " 25×8 " be equal to "200 foot-pounds"?

Likewise, the expression "Horse power = $\frac{\text{Work done in foot pounds}}{33,000 \times \text{time in minutes}}$ ", as a scientific formula, is open to criticism. If by "Horse power" we mean

"33,000. $\frac{\text{ft. lbwt.}}{\text{min}}$," or "33,000. ft. lbwt per min", the equality has no meaning. If we replace the words "Horse power" by "Number of Horse power," the expression again has no meaning, for on the right-hand side we have a number of $\frac{\text{"foot pounds"}}{\text{min.}}$. We should use either $\text{Power} = \frac{\text{Work}}{\text{Time}}$ or $h = \frac{w}{33,000t}$ where $\text{Work} = w$. ft. lbwt, $\text{Time} = t$. min., $\text{Power} = h$. horse power $= h \cdot 33,000 \cdot \frac{\text{ft. lbwt}}{\text{min}}$. The analytical geometry is well treated. The rule for locating the bisectors of angles is simply stated.

Part of the calculus section is weak; a more precise definition of "definite integral" would have increased the value of the book: "definite integral" should not be defined in terms of "differentials."

The examples are well chosen and nicely graded; they are not poached from the preserves of applied mechanics, applied heat, etc.

In the preface, the authors state that the course is intended for use in the freshman year of colleges and universities. Apart from the blemishes referred to, the book could be usefully employed in this country for post-Matriculation work in a Secondary school, or as a practical text-book in a Technical school.

Its clear and concise style marks it out as a book to be recommended for this purpose.

V. NAYLOR.

761. "What concerns us . . . is . . . Laplace's interview with Napoleon. The story goes that the monarch asked the astronomer what room there was for God in his 'celestial mechanics.' Laplace replied that he 'had no need of that hypothesis,' and this answer has been terribly misunderstood. In the first place, we cannot for a moment imagine that there was any lack of seriousness on Laplace's part. No one could be flippant to Napoleon! In the second place, we may be sure that Laplace was not making a profession of atheism. What Laplace meant was that no one could speak of God and of Gravitation in the same breath. . . . Laplace meant that the august concept of God is foreign to the astronomer's 'universe of discourse,' as the philosophers say."—*The Gospel of Evolution*, by Prof. J. Arthur Thomson, ch. i. pp. 24-25.

762. "Our search after the ideal to which we apply the august word TRUTH is asymptotic; that is to say, it is like a curve which is always approaching a right line but never reaching it; . . . 'The human mind, born too great for its ends, never at peace with its goal, is doomed ever to seek for that which it has not the power to attain.' Man has a passion for the asymptotic."—*Loc. cit.* ch. vii. pp. 174-175.

763. "We need not boggle over the question, If the universe is finite, what is beyond? So might a two-dimensional caterpillar, burrowing in the skin of an orange, say to itself: 'If I go on and on, I *must* come to an end sometime.' We are tri-dimensional caterpillars—all except some mathematicians, who are quite at home in the four-dimensional curved world, with Time and Space interlaced."—*Loc. cit.* ch. vii. p. 178.

764. "[The Banians (traders of Gujerat)] are generally good arithmeticians, till of late have little else than number of the mathematics save in the art of dialling; concerning which some report that the Banians here had a clock that struck 64 times in 24 hours. The day and night they divide into four, and subdivide that into eight; and some little skill they have in navigation."—Thomas Herbert, *Travels in Persia*, 1627-1629, ed. Foster (1929), p. 36. [Per Mr. Puryer Whyte.]

765. In the highest sense, namely that of an accurate thinker, Faraday was, as has been often said, a great mathematician, although he was so little of a mathematical expert that he once expressed his obligation to one who had calculated the tangents of some galvanometer deflections for him.—G. Chrystal, Review of J. C. Maxwell's *Electricity and Magnetism*, *Nature* 25, p. 237.

CORRESPONDENCE.

HIGHER TRIGONOMETRY FOR SCHOOLS.

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—Professor Carslaw's review of Siddons and Hughes' *Trigonometry*, Parts III and IV, shows such a lack of understanding of the problem of school teaching that I am moved to write to you and to urge that the needs of school teaching should get a little more help and sympathy from the *Gazette*.

A man who is used to dealing with university students will naturally treat analysis with extreme rigour. A man who has to deal with boys knows that such a treatment for beginners in the study of analysis would produce no results at all except with boys of very exceptional mathematical ability, and that with such boys it is nearly always better to point out the difficulties as they occur, but to pass them by for the moment.

Just as it is absurd with beginners in geometry to approach the subject with all the rigour which men like Hilbert, Peano, Whitehead and Russell have reached, so most teachers of boys will agree that their first introduction to infinite series and functions of complex numbers should not aim at the standard of rigour of books such as Bromwich's *Infinite Series*. The history of the subject is some guide to the teaching of it; the rigour of to-day is a modern growth and is not suitable for a boy who is beginning the subject—he cannot appreciate it because to him it is meaningless until he has handled some infinite series and functions of complex numbers.

A reviewer of a book should consider for whom the book is written and should not judge a book written for schoolboys by the standard he expects from those same boys a few years later after a university course.

We have tried to make it clear in this book that we did not attempt to produce a book that would satisfy the tests of modern analysis, but we set out to give the boy a preliminary canter round part of the course, pointing out its difficulties and pitfalls, and to prepare him for the hard work he would have to do when he comes to cover it under university conditions, or even, in the case of exceptional boys, later in his school career. Any teacher of boys knows the value of a preliminary survey of a difficult subject. To attempt strict rigour with beginners would only choke them off altogether. Professor Carslaw seems to question the need or wisdom of including such a survey. But, apart from the fact that questions on these subjects are set in Entrance Scholarship Examinations, I am convinced that such a course is desirable; students at the university who have gone through such a course at schools say how much it has helped them and also say that, without it, the lectures at the university would have been beyond them.

For many years schoolmasters have wanted a book on Higher Trigonometry that paved the way for boys to go on to the more exact treatises and the lectures that they would meet at the universities. One essential of such a book seemed to be that it should be honest and point out that there were difficulties that would need to be investigated at a later stage.

Many books are written which do not even point out where they are making assumptions that need justification. On page 303 we say, "In this book we do not propose to deal thoroughly with the question of convergence... because it seems best for a student to study the subject *after* he has had some acquaintance with infinite series; on the other hand we shall try to point out places at which we make assumptions about convergence and at which difficulties arise." Lower down on the page we advise that the student should go on after reading this book to study the subject in books that deal with it rigorously. Also we frequently refer to the dangers and pitfalls in the use of complex numbers (see pp. 289, 300).

Professor Carslaw is quite right in saying that our statements about differentiation and integration are careless in detail. They do not apply even to power series, and it should have been said that no criterion applicable to other series could be so much as enunciated within the range of our treatment of convergence. But, on the general principle of acquiring some familiarity with a process before discussing its justification, I am entirely unrepentant.

I contend that a boy who goes to the university able to differentiate or integrate the terms of a series and also the sum of the series, and with the knowledge that the two results may be equated in some cases, but with a clear understanding that the equating of the two results is a matter that needs investigation, is better prepared to undertake that investigation than one who has been taught to equate the two results without any warning, or than one who has never even thought of the possibility of using differentiation or integration in connection with series. Is Professor Carslaw fair to us when he says, "Bromwich in § 60 pointed out that the 'proof' which starts with the assumption that $\sin x$ and $\cos x$ can be expressed as Power Series is not logically complete. Siddons and Hughes, on the other hand, consider the 'Proof' quite sound"? In the book under review we state quite clearly, "We shall assume that $\sin \theta$ can be expressed as a series of powers of θ " (page 252). That is an honest statement, and any impartial judge must acknowledge that we have made it clear that our 'proof' depends on that assumption.

Professor Carslaw suggests that we should have used Tannery's theorem to obtain the infinite products for $\sin x$, etc., and his interesting article on all this work appears in the *Mathematical Gazette* (p. 71); but does he imagine that it is suitable for a first treatment for a boy who has had no introductory course?

In spite of all that Professor Carslaw says there are other points of view, and many practical teachers encourage me to think that the book presents a most difficult subject in a form intelligible to beginners, and honestly and fairly points out assumptions and difficulties, while treating examples where possible with a rigour not approached in any other text-books at present possible for school use.

But to return to my original plea: will the *Mathematical Gazette* help us by giving us more articles on actual school teaching? I can well imagine the editors saying, "Write such articles and we will publish them." But who will venture to write such articles if they are going to be criticised by the standard of modern analysis by men who seem to be entirely out of sympathy with school teaching?

Our parent Association, the A.I.G.T., was founded by schoolmasters for the benefit of mathematical teaching in schools. I have before me a copy of the first report of that Association. That report gives the list of members, 61 in all: of those, 51 were schoolmasters. To-day I suppose the majority of the members of the Mathematical Association are engaged in teaching in schools; they do not begrudge the space given to the many admirable reviews by eminent mathematicians of books of extreme mathematical rigour, but they would like to have all books intended for school purposes reviewed from the school point of view by men who understood what are the needs of the schools, and not treated as though they were intended for university students.

My letter has already run to great length, but I must say that it is not all university professors who are blind to the school problem; I should like to pay my tribute to the great help school mathematics has received from some of the most eminent of pure mathematicians: they have displayed a wonderful understanding of the problem of teaching in schools, they have appreciated the fact that the rigour required of university students is not to be expected of the schoolboy beginning the study of pure mathematics, and that the boy

must be led gradually to strict rigour. Such men have given a helping hand to many schoolmasters and so to many boys who have become good mathematicians; but the men who think that the mere schoolboy should be fed from the start with mathematical food of the strictest rigour are doing their best entirely to stop the flow of promising mathematicians to the universities; if schoolmasters followed their advice, the number of boys choosing mathematics as their special subject would soon tend to the limit zero.

Finally, I would suggest that the *Mathematical Gazette* should throw open its pages to a discussion of the following three possible ways of dealing with Higher Trigonometry in schools:

(a) A treatment that is rigorous from the start.

(b) A first course that does not pretend to be rigorous and makes many assumptions that need justification, but points out the assumptions and dangers and pitfalls to be investigated later.

(c) The old-fashioned course which gives "proofs" that make assumptions that are not stated, and slurs over difficulties and pitfalls without any warning (e.g. proofs of the power series for $\sin x$ and $\cos x$ that neglect an infinite number of infinitely small quantities in the most light-hearted fashion).

My answer would be

(a) The book has yet to be written that will make this possible. And will it ever be possible except with very few boys? And how many schoolmasters are capable of such a treatment? Even if they know all about the work, they so seldom get a boy of that class that they cannot have much experience of teaching it, and it would be better for them to leave the work to the university teacher.

(b) is to me the best course. I believe the really larger-minded university teachers will approve of it and see that it paves the way for their work; the boy will come to them with a knowledge of the ground and prepared to face the difficulties.

(c) which is the course books have catered for in the past, seems to me thoroughly bad. Boys who have been trained on that line go to the university very ill prepared—they do not even see that there are difficulties to be faced.

A symposium on this would be valuable, and I hope that other schoolmasters will express their views and not be frightened by the fear of "high brow" criticism from men who have not had school experience.—Yours, etc.,

A. W. SIDONS.

P.S.—I do not wish to imply that criticism from men with university experience is valueless; it will be very helpful, provided it has some sympathy with the conditions that obtain in schools of various types.

766. "During the three years which I spent at Cambridge, my time was wasted. . . I attempted mathematics, . . . but I got on very slowly. The work was repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. . . In my last year I worked with some earnestness for my final degree of B.A., and brushed up my Classics, together with a little Algebra and Euclid, which latter gave me much pleasure. . . The logic of Paley[*'s Evidences and Natural Theology*] gave me as much delight as did Euclid. . . I was very intimate with Whitley, who was afterwards Senior Wrangler. . . I also got into a musical set, I believe by means of my warm-hearted friend, Herbert, who took a high wrangler's degree."—From *Autobiography of Charles Darwin*, pp. 21, 22, 23, 24. [Whitley, Rev. C., afterwards Canon of Durham. Herbert, John Maurice, afterwards County Court Judge of Cardiff and Monmouth Circuit.]

PERSONAL NOTES.*

Members of the Association have the pleasure of congratulating Prof. Sir T. P. NUNN, M.A., D.Sc., on the knighthood bestowed upon him in recognition of his services to education.

Dr. P. A. M. DIRAC, of St. John's College, Cambridge, has been elected a Fellow of the Royal Society. Dirac's q -numbers, Heisenberg's *Matrix Mechanics* and Schrödinger's *Wave Mechanics* are three aspects of what is called *The New Quantum Theory*, perhaps the most rapidly growing branch of modern mathematical physics.

Dr. CARL STØRMER, Professor of Pure Mathematics in the University of Oslo (Norway), the leading authority on the aurora polaris and magnetic storms from both the observational and mathematical aspect, is delivering a series of lectures on these subjects in London, Cambridge, Oxford, Manchester, Edinburgh, and Aberdeen.

Dr. R. A. FISHER, F.R.S., of the Rothamsted Experimental Station, will lecture on Thursday afternoons, 1st May to 12th June, at The Imperial College, South Kensington, on advanced Statistical Methods. It is interesting to notice the extensive use now being made of mathematical methods at a biological research institute.

Mr. A. E. INGHAM, at present Reader in Mathematical Analysis at the University of Leeds, has been appointed Lecturer and Director of Mathematical Studies at King's College, Cambridge.

Mr. W. R. DEAN, of Trinity College, Cambridge, has been appointed a University Lecturer in Mathematics.

Dr. A. T. DOODSON, Associate Director of the Liverpool Observatory and Tidal Institute, has been awarded the Gray Navigation Prize of the Society of Arts for his work on the analysis and prediction of tidal currents.

The University of London has conferred the degree of D.Sc. on Mr. W. G. BICKLEY for a thesis entitled "Two-dimensional potential problems concerning a single closed boundary," and other papers.

On 2nd December, 1929, was commenced the building of a new Mathematical Institute at the University of Göttingen. This realisation of the ideals of FELIX KLEIN was rendered possible by the generous gift of the Rockefeller Foundation.

The Third International Congress of Technical Mechanics will be held at Stockholm, 24th to 29th August, 1930. The Secretary-General is Prof. W. WEIBULL, Ecole Technique Supérieure, Valhallavägen, Stockholm, Sweden.

Research prizes in the University of Cambridge have been awarded as follows: Smith's prizes to R. E. A. C. Paley and J. A. Todd, Rayleigh prizes to W. R. Andress and L. C. Young, all of Trinity. Mr. Todd's paper on Four-dimensional Space appeared in the March number of the *Proc. of the L.M.S.* Mr. L. C. Young is the author of the Cambridge Tract "The Theory of Integration." He belongs to a mathematical family. His father is Prof. W. H. Young, F.R.S.; his mother is a former pupil of Felix Klein, and was the first woman to take a Ph.D. in Prussia, while his sister, Miss Rosalind C. Young, has also published researches on integration.

767. Question. Prove that if a polynomial $f(x)$ is divided by $x - a$ until the remainder is free of x , that remainder is $f(a)$.

Answer.

$$\begin{array}{rcl} x - a) f(x) & & (f \\ & \underline{f(x) - f(a)} & \\ & f(a) & \end{array}$$

Q.E.D.

* The Editor will be glad to receive early intimation of appointments of Mathematical Teachers at the leading schools, and of distinctions, etc., such as are likely to be of general interest to our readers.

THE LIBRARY.

160 CASTLE HILL, READING.

The Librarian reports gifts as follows :

From Prof. **R. C. Archibald**, offprints of his papers, and his
Bibliography of Egyptian Mathematics, with Supplement - 1927, 1929

Reprinted from the edition of the Rhind Papyrus prepared by A. B. Chace, L. S.
Bull, H. P. Manning, and R. C. Archibald.

From Prof. **A. S. Eddington** :

C. HUYGENS *Euvres Complètes.* XVI - - - - - 1930

From Miss **M. A. Graves**, text-books by W. H. Besant, F. Castle (2), G. A. Gibson, E. H. Griffiths, H. S. Hall (2), R. B. Hayward, J. B. Lock, A. Lodge, S. L. Loney, G. Salmon, F. W. Sanderson, F. H. Stevens (2).

From Mr. **W. J. Greenstreet** :

H. BROGGI *Análisis Matemático.* II - - - - - 1927

Vol. I would be very welcome.

R. GANS *Cálculo Vectorial* - - - - - 1926

R. G. LOYARTE *Física General.* I (2) - - - - - 1927

J. MACLEAN *Graphs and Statistics* - - - - - 1926

A. N. WHITEHEAD and **B. A. W. RUSSELL**
Principia Mathematica. II-III

(2, i.e. 1 (1912, 1913) rep.) 1927

*Vol. I, which was substantially altered, would be a valuable
addition to the Library in either edition.*

From Mr. **A. M. Grundy** :

L. BIANCHI *Differentialgeometrie* (2) - - - - - 1910

Translated from Italian into German by M. Lukat.

A. CAYLEY *Elliptic Functions* - - - - - 1876

H. DURÈGE *Elliptische Functionen* (3) - - - - - 1878

Theorie der Funktionen (5 : L. Maurer) - - - - - 1908

Maurer, like Fiedler, is rather a part-author than a simple editor.

L. P. EISENHART *Differential Geometry* - - - - - 1909

A. R. FORSYTH *Differential Geometry* - - - - - 1912

H. HILTON *Homogeneous Linear Substitutions* - - - - - 1914

C. F. KLEIN *Lectures on the Icosahedron* (2) - - - - - (1913)

Translated from German by G. G. Morrice.

R. V. LILIENTHAL *Differentialgeometrie.* I; II i - - - - - 1908, 1913

ALL PUBLISHED.

G. SALMON *Analytic Geometry of Three Dimensions*
(5 : R. A. P. Rogers). II - - - - - 1915

*Has any member the first volume of this edition to spare ?
There are two earlier one-volume editions in the Library.*

G. SALMON und W. FIEDLER

Analytische Geometrie der Kegelschnitte

(2 vols.) (7, 6) - - - - - 1907, 1903

Analytische Geometrie der Höheren Ebenen Kur-
ven (2) - - - - - 1882These German works are adapted with considerable freedom
from the English originals.

O. SCHLOEMILCH Intégrales et Fonctions Elliptiques - - - 1873

Translated from German into French, and provided with
an introduction on the theory of functions, by J. Graindorge.From Mr. C. W. Payne, text-books by P. André, T. Muir, R. C. J. Nixon,
R. Potts, C. Taylor, J. P. Zanen (2), together with :

W. H. DREW Geometrical Conic Sections (7) - - - - - 1883

A. HENRY Calculus and Probability - - - - - 1922

For actuarial students.

H. HILTON Groups of Finite Order - - - - - 1908

O. J. LODGE Easy Mathematics chiefly Arithmetic - - - 1905

From Mr. A. S. Percival, an offprint and the following of his books :

Geometrical Optics (1 (1913) rep.) - - - - - 1925

Perspective - - - - - 1921

Prescribing of Spectacles (3) - - - - - 1928

Also from Miss M. A. Graves, Miss M. J. Parker, Mr. C. W. Payne, Miss
J. M. Robertson, Prof. J. E. A. Steggall, and Miss C. M. Waters, collections
of back numbers of the *Gazette*.

The following have been bought :

M. BRÜCKNER Vielecke und Vielfache - - - - - 1900

The sequel was given to the Library in 1924.

R. D. CARMICHAEL Diophantine Analysis - - Merriman-Woodward 16 1915

G. W. HEARN Researches on Curves of the Second Order, also on
Cones and Spherical Conics - - - - - 1846

M. E. C. JORDAN Cours d'Analyse. II : Calcul Intégral (2) - - 1894

*Vol. III is still wanted to complete the set.*G. KIRCHHOFF Mathematische Physik. II : Optik ; III : Elec-
tricität und Magnetismus - - - - - 1891Posthumous volumes edited by K. Hensel and M. Planck
respectively.
Completing the set of this Treatise in the Library.

M. LECAT Bibliographie du calcul des variations, 1850-1913 - 1913

T. LEYBOURN Mathematical Questions from the *Ladies' Diary*
(4 vols.) - - - - - 1817To gauge the contemporary interest in the *Diaries*, notice
that these volumes were published at £4 the set.

J. McMAHON Hyperbolic Functions (4) Merriman-Woodward 4 - 1906

This is the first issue as a separate monograph, but the three
editions of the volume of which this originally formed a
chapter have been taken into count on the title-page.

D. E. SMITH

History of Modern Mathematics (4)

Merriman-Woodward 1 1906

The reckoning of the edition is the same as for McMahon's monograph.

DUKE OF SOMERSET

Alternate Circles and their connexion with the
Ellipse

1850

Elementary Properties of the Ellipse (2)

1843

THE ANALYST.

The complete run, from 1874 to 1883 in 10 volumes, of the precursor of the *Annals of Mathematics*, of which the Library has a perfect set.

THE MATHEMATICAL VISITOR.

One of the journals brought into existence and maintained by that lopsided enthusiast Artemas Martin, who was not only editor and publisher, but compositor also. The contents consist almost entirely of problems and solutions, and although the demand for the first number evoked a second edition, the journal died a lingering death: the first volume, in six numbers, covers 1877 to 1881, and the dates attached to the four numbers of the unfinished second volume are 1882, 1883, 1890, 1895.

MEMOIRS OF THE ANALYTICAL SOCIETY.

The volume for 1813 was the only one issued.

NOUVELLES ANNALES DE MATHÉMATIQUES.

The second series, 1862-1881, complete in 20 volumes. The Library has the first series, and a number of later volumes, as well as a few duplicates which could be exchanged to fill gaps.

NOUVELLE CORRESPONDANCE MATHÉMATIQUE.

The set of six volumes, 1875-1880.

ERRATUM.

Vol. xv. p. 32, l. 8 up. *For Macaulay's read Macaulay*

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